Groundwork

In this chapter we lay the foundation for the computation of the trace of $T_n$. First we show that $S_k(N, \omega')$ embeds isometrically into $L^2_0(\omega)$ for an appropriately chosen Hecke character $\omega$. This having been done, the main result of this chapter is the construction of a function $f$ on $G(\mathbf{A})$ for which the following diagram commutes:

$$
\begin{array}{c}
L^2(\omega) \xrightarrow{n^2-1R(f)} L^2(\omega) \\
\text{orthog. proj.} \downarrow \quad \quad \downarrow \\
S_k(N, \omega') \xrightarrow{T_n} S_k(N, \omega')
\end{array}
$$

In particular, although $L^2(\omega)$ is infinite-dimensional, $n^2-1R(f)$ will be an operator of finite rank (with $\text{rank}_{\mathbb{C}} R(f) \leq \dim_{\mathbb{C}} S_k(N, \omega')$) and having the same eigenvalues as $T_n$.

12. Cusp forms as elements of $L^2_0(\omega)$

12.1. From Dirichlet characters to Hecke characters. Let $\omega'$ be a Dirichlet character modulo $N$ satisfying (3.12):

$$\omega'(-1) = (-1)^k.\tag{12.1}$$

Using strong approximation for the ideles

$$\mathbf{A}^* = \mathbb{Q}^*(\mathbb{R}_+^* \times \hat{\mathbb{Z}}^*),$$

we use $\omega'$ to define a Hecke character of $\mathbf{A}^*$ (trivial on $\mathbb{Q}^*$ and $\mathbb{R}_+^*$):

$$\omega : \mathbf{A}^* \longrightarrow \hat{\mathbb{Z}}^* \longrightarrow (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow \mathbb{C}^*,\tag{12.2}$$

where the first arrow is the canonical projection, the second arrow is the quotient map, and the last arrow is given by $\omega'$. Let

$$\pi_N : \prod_{p|N} \mathbb{Z}_p \longrightarrow \mathbb{Z}/N\mathbb{Z}$$

be the canonical surjection. For any idele $x \in \mathbf{A}^*$, let $x_N$ be the idele which agrees with $x$ at the places $p|N$, and which is 1 at all other places. Then

$$\text{for } x \in \mathbb{R}_+^* \times \hat{\mathbb{Z}}^*, \quad \omega(x) = \omega(x_N) = \omega'(\pi_N(\prod_{p|N} x_p)).\tag{12.3}$$
If $d$ is an integer coprime to $N$, then $d_N \in R_+^* \times \hat{Z}^*$, so by the above,

\begin{equation}
\omega(d_N) = \omega'(d).
\end{equation}

(However $\omega(d) = 1$ since $d \in Q^*$). More generally, if $d$ is an arbitrary nonzero integer, it is not hard to check that

$$\omega(d_N) = \omega'(\prod_{p|N} p^{v_p(d)}).$$

The above procedure can be reversed. A Hecke character $\omega$ has finite order if there exists an integer $\ell$ such that $\omega(x)^\ell = 1$ for all $x \in A^*$. Such a character is necessarily unitary.

**Lemma 12.1.** A Hecke character has finite order if and only if it is unitary and trivial on $R_+^*$.

**Proof.** Suppose $\omega$ has order $\ell \geq 1$. Define $\omega' : R_+^* \rightarrow C^*$ by $\omega'(x) = \omega(x_\infty \times 1_{\text{fin}})$. Such a character must be of the form $\omega'_\infty(x) = x^s$ for some $s \in C$ by Proposition 11.6. Now $\omega'_\infty(x)^\ell = x^{s\ell} = 1$ for all $x \in R_+^*$, so we must have $s = 0$. Thus $\omega$ is trivial on $R_+^*$.

Conversely, suppose $\omega$ is a (unitary) Hecke character which is trivial on $R_+^*$. Then $\omega$ defines a continuous homomorphism $\omega : \hat{Z}^* \rightarrow C^*$. If $O \subset C^*$ is a small open neighborhood of 1, then $\omega^{-1}(O)$ is open, and hence contains $U_M \subset \hat{Z}^*$ for some $M > 0$. Then $\omega(U_M) \subset O$ is a subgroup of $C^*$, which must be trivial if $O$ is sufficiently small. Thus each such $\omega$ factors through $\hat{Z}^*/U_M \cong (Z/MZ)^*$ for some positive integer $M$. \hfill $\Box$

If in the above proof $M > 0$ is chosen to be as small as possible, we set $N_\omega = M$ and call this integer the conductor of $\omega$. In this way, there is a natural bijection

$$\left\{ \text{Dirichlet characters of conductor } M \right\} \leftrightarrow \left\{ \text{finite order Hecke characters of conductor } M \right\}.$$ 

We remark that for the character defined in (12.2),

$$N_\omega = N_{\omega'}.$$ 

The character $\omega'$ may not be primitive, i.e. $N$ may not be minimal, so we can only say that $N_\omega | N$.

A continuous character of $Q_+^*$ is unramified if its kernel contains $Z_p^*$. Every continuous character of $A^*$ factorizes as a product of local characters, all but finitely many of which are unramified. Let $\omega$ be the character defined in (12.2). We factorize $\omega$ in this way as follows. For $x_p \in Q_p$ ($p \leq \infty$), define

$$\omega_p(x_p) \overset{\text{def}}{=} \omega(1, \ldots, 1, x_p, 1, 1, \ldots).$$
For \( p \) finite, suppose \( v_p(x_p) = j \) so that \( x_p = p^j u \), where \( u \in \mathbb{Z}_p^* \). Then if \( p \nmid N \),
\[
(12.5) \quad \omega_p(x_p) = \omega(p^j (p^{-j}, \ldots, p^{-j}, u, p^{-j}, \ldots)) = \omega'(p)^{-j}.
\]
In particular, if \( j = 0 \) then \( \omega_p(u) = 1 \), so \( \omega_p \) is unramified when \( p \nmid N \). As a result, the following decomposition holds for any \( x \in \mathbb{A}^* \):
\[
\omega(x) = \prod_{p \leq \infty} \omega_p(x_p).
\]
Using (12.1) and (12.3), it is easy to show that
\[
(12.6) \quad \omega_{\infty}(x) = \text{sgn}(x)^k.
\]
Suppose \( d > 0 \) and \( \gcd(d, N) = 1 \). Then \( \omega_{\infty}(d) = 1 \) and \( \omega_p(d) = 1 \) for all \( p \nmid dN \) since \( \omega_p \) is unramified. Therefore
\[
1 = \omega(d) = \prod_{p \mid d} \omega_p(d) \prod_{p \mid N} \omega_p(d) = \prod_{p \mid d} \omega_p(d) \omega'(d)
\]
by (12.4). Thus
\[
(12.7) \quad \prod_{p \mid d} \omega_p(d) = \omega'(d)^{-1} \quad (d > 0, (d, N) = 1).
\]

12.2. From cusp forms to functions on \( G(A) \). We now review the embedding
\[
S_k(N, \omega') \longrightarrow L^2_{\infty}(\omega).
\]
Recall that we have defined \( K_0(N) = \prod_{p < \infty} K_0(N)_p \), where
\[
K_0(N)_p = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_p \mid c \equiv 0 \mod N \right\}.
\]
By strong approximation for \( G(A) \), we have
\[
(12.8) \quad G(A) = G(\mathbb{Q})(G(\mathbb{R})^+ \times K_0(N)).
\]
If \( k_0 = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_0(N) \), define
\[
(12.9) \quad \omega(k_0) = \omega(dN).
\]
Because \( \omega(dN) \) depends only on \( dN \) modulo \( N\hat{\mathbb{Z}}^* \), this defines a character of \( K_0(N) \). If \( k_0 = \gamma \in \Gamma_0(N) \subset K_0(N) \), then by (12.4) this agrees with \( \omega'(\gamma) \) defined in (3.9):
\[
(12.10) \quad \omega(\gamma) = \omega(dN) = \omega'(d) = \omega'(\gamma).
\]
By identifying \( Z(A) \) with \( \mathbb{A}^* \), we also view \( \omega \) as a character of \( Z(A) \). Suppose \( z \in Z(A) \), and write \( z = z_{\mathbb{Q}}(z_{\infty} \times z_0) \) with \( z_{\infty} \in Z(\mathbb{R})^+ \), and \( z_0 \in Z(\hat{\mathbb{Z}}) \subset K_0(N) \). Then by (12.3),
\[
(12.10) \quad \omega(z) = \omega((z_{\infty} \times z_0) N) = \omega(z_0),
\]
where \( \omega(z_0) \) is defined as in (12.9).
For \( z \in Z(A_{\text{fin}}) \) define \( \omega(z) = \omega(1_{\infty} \times z) \). Then we find by the above that for

\[
(12.11) \quad z \in Z(A_{\text{fin}}) \cap K_0(N),
\]

the value of \( \omega(z) \) is independent of whether we regard \( z \) as belonging to \( Z(A_{\text{fin}}) \) or \( K_0(N) \).

There is a chart in the Appendix which tabulates the various uses of \( \omega \) and \( \omega' \).

For \( h \in W_k(N, \omega') \), define a left \( G(Q) \)-invariant function \( \phi_h \) on \( G(A) \) by

\[
(12.12) \quad \phi_h(\gamma(g_{\infty} \times k_0)) = \omega(k_0)j(g_{\infty}, i)^{-k_h(g_{\infty}(i))},
\]

for \( \gamma \in G(Q) \), \( g_{\infty} \in G(R)^+ \), and \( k_0 \in K_0(N) \). The decomposition

\[
g = \gamma(g_{\infty} \times k_0)
\]

is not unique, so we will show that \( \phi_h(g) \) is well-defined. Suppose \( g = \gamma'(g'_{\infty} \times k'_0) \) is another decomposition. Then

\[
\gamma'^{-1}\gamma = g'_{\infty}g_{\infty}^{-1} \times k'_0k_0^{-1} \in (G(R)^+ \times K_0(N)) \cap G(Q) = \Gamma_0(N).
\]

Thus any decomposition of \( g \) has the form \( g = \gamma\delta^{-1}(\delta_{\infty}g_{\infty} \times \delta_{\text{fin}}k_0) \) for some \( \delta \in \Gamma_0(N) \). To check that \( \phi_h \) is well-defined, we must check that inserting \( \delta \) in this manner does not affect the value of \( \phi_h \). We have

\[
\phi_h(\gamma\delta^{-1}(\delta_{\infty}g_{\infty} \times \delta_{\text{fin}}k_0)) = \omega(\delta_{\text{fin}}k_0)j(\delta g_{\infty}, i)^{-k_h(\delta g_{\infty}(i))} = \omega(k_0)j(g_{\infty}, i)^{-k_h(g_{\infty}(i))} = \phi_h(\gamma(g_{\infty} \times k_0)),
\]

as needed.

If \( g = \gamma(g_{\infty} \times k_0) \in G(A) \) and \( z = zQ(z_{\infty} \times z_0) \), then

\[
(12.13) \quad \phi_h(zg) = \omega(z_0k_0)j(z_{\infty}g_{\infty}, i)^{-k_h(z_{\infty}g_{\infty}(i))} = \omega(z)\phi_h(g),
\]

by equations (12.10), (3.7) and (3.8). Thus \( \phi_h \) has central character \( \omega \), and our goal is to show that \( \phi_h \in L^2_\theta(\omega) \) when \( h \) is a cusp form.

12.3. Comparison of classical and adelic Fourier coefficients. Let \( h \in W_k(N, \omega') \). Fix any \( g \in G(A) \) and consider the map \( A \to C \) defined by

\[
x \mapsto \phi_h\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right) g.
\]

By the \( G(Q) \)-invariance of \( \phi_h \) this defines a continuous function on \( Q \setminus A \). Therefore by Proposition 8.10 it has a Fourier expansion

\[
(12.14) \quad \phi_h\left(\begin{array}{c} 1 \\ x \\ 1 \end{array}\right) g = \sum_{\beta \in Q} W_\beta(g) \theta(-\beta x),
\]

assuming absolute convergence of the series. The Fourier coefficient \( W_\beta(g) \) is called the \( \theta_\beta \)-Whittaker function of \( \phi_h \). Our goal in the next proposition is
to compute $W_\beta(g)$ explicitly. In fact the above Fourier expansion is closely related to the Fourier expansion of $h$ at a certain cusp determined by $g$. Consequently, we will see that (12.14) is justified, and we will be able to prove that $\phi_h \in L^2_0(\omega)$ when $h$ is a cusp form.

First we claim that for computing $W_\beta(g)$, it suffices to consider the case where $\det g \infty > 0$. In fact, suppose $g' = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} g$. Then 

$$W_\beta(g) = W_{-\beta}(g').$$

Indeed, because $\phi_h$ is $G(\mathbb{Q})$-invariant,

$$\phi_h(ng) = \phi_h \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} g' \right) = \phi_h \left( \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} g' \right) = \phi_h(n^{-1}g').$$

Therefore

$$W_\beta(g) = \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \phi_h(ng) \theta_\beta(n) dn = \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \phi_h(n^{-1}g') \theta_{-\beta}(n^{-1}) dn$$

$$= W_{-\beta}(g'),$$

as claimed. Here (and below) we regard $\theta_\beta$ as a character on $N(\mathbb{Q}) \setminus N(\mathbb{A})$ in the obvious way.

**Proposition 12.2.** Let $h \in W_k(N, \omega')$, and let $\phi_h$ be the associated function defined in (12.12). For $\delta \in G(\mathbb{Q})^+$ write

$$h_\delta(z) = \sum_{n \in \mathbb{Z}} a_n(\delta) q^n,$$

where $q = e^{2\pi i z}$ and $M = M_2(\Gamma_1(N))$ is given in Lemma 3.7. Fix $g \in G(\mathbb{R})^+ \times G(\mathbb{A}_{\text{fin}})$, and consider the Fourier expansion (12.14). Then there exists $\delta \in G(\mathbb{Q})^+$, determined by $g_{\text{fin}}$ in (12.17) below, such that for any $\beta \in \mathbb{Q}$,

$$W_\beta(g) = \begin{cases} j(g_\infty, i)^{-k} e^{-\frac{2\pi iz}{M}} a_n(\delta) & \text{if } \beta = \frac{n}{M} \in \frac{1}{M} \mathbb{Z} \\
0 & \text{if } \beta \notin \frac{1}{M} \mathbb{Z}, \end{cases}$$

where $z = g_\infty(i)$. (If $g_{\text{fin}} \in G(\mathbb{Q})^+$, then $\delta = g_{\text{fin}}^{-1}$.) In particular, taking $\beta = 0$, we see that

(12.15) $$W_0(g) = (\phi_h)_N(g) = j(g_\infty, i)^{-k} a_0(\delta)$$

is the constant term of $\phi_h$.

Before proving the proposition, we highlight two consequences.

**Corollary 12.3.** In the above notation, $\sum |W_\beta(g)| < \infty$, so (12.14) is justified.

**Proof.** Let $z = g_\infty(i)$ and $q = e^{2\pi iz/M}$. By the proposition,

$$\sum_{\beta \in \mathbb{Q}} |W_\beta(g)| = |j(g_\infty, i)|^{-k} \sum_{n \in \mathbb{Z}} |a_n(\delta)q^n|,$$

which is finite since $h_\delta(q) = \sum a_n(\delta)q^n$ is absolutely convergent. $\Box$
Corollary 12.4. Let $h \in S_k(N, \omega')$, and write $h(z) = \sum_{n>0} a_n q^n$, where $q = e^{2\pi i z}$. Then for $m \in \mathbb{Q}$,

$$
\int_{\mathbb{Q} \backslash \mathbb{A}} \phi_h\left(\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}\right) \theta(mt) dt = \begin{cases} e^{-2\pi m} a_m & \text{if } m \in \mathbb{Z}^+ \\ 0 & \text{otherwise}. \end{cases}
$$

Proof. Apply the proposition with $g = 1$, so $\delta = 1$, $M = 1$, $\beta = m$, and $z = i$.\hfill \Box

Proof of the proposition. Let

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \mid d \equiv 1 \mod N\hat{\mathbb{Z}} \right\}.
$$

Using (12.12) it is immediate that

(12.16) $\phi_h(gk) = \phi_h(g)$

for all $k \in K_1(N)$.

Note that $\det K_1(N) = \hat{\mathbb{Z}}^\ast$, so by strong approximation we can write

(12.17) $g_{\text{fin}} = \delta^{-1} k$

for some $\delta \in G(\mathbb{Q})$ and $k \in K_1(N)$. Multiplying both $\delta$ and $k$ by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ if necessary, we can assume $\delta \in G(\mathbb{Q})^\ast$. Let $z = g_{\infty}(i)$. Then using (12.16),

(12.18)

$$
\phi_h(g_{\infty} \times g_{\text{fin}}) = \phi_h(g_{\infty} \times \delta^{-1} k) = \phi_h(\delta g_{\infty} \times 1_{\text{fin}})
$$

$$
= j(\delta g_{\infty}, i)^{-k} h(\delta(z))
$$

$$
= j(g_{\infty}, i)^{-k} j(\delta, z)^{-k} h(\delta(z)) = j(g_{\infty}, i)^{-k} h(\delta(z)).
$$

Let $M = M_\delta(\Gamma_1(N))$ be the positive rational number given in Lemma 3.7 (page 15). By definition, $M$ is the positive rational number satisfying

$$
N(\mathbb{Q}) \cap \delta^{-1} \Gamma_1(N) \delta = N(M\hat{\mathbb{Z}}) = \left\{ \begin{pmatrix} 1 & tM \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z} \right\}.
$$

Thus

$$
\delta \left\{ \begin{pmatrix} 1 & tM \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Z} \right\} \delta^{-1} \subset \Gamma_1(N).
$$

In particular, the lower left entry, as a linear function of $t$, is congruent to $0 \mod N$, and the lower right entry is congruent to $1 \mod N$. This remains true if we allow $t$ to range through all of $\hat{\mathbb{Z}}$ instead of $\mathbb{Z}$. Consequently,

$$
\delta N(M\hat{\mathbb{Z}}) \delta^{-1} \subset K_1(N).
$$

For $n \in N(A)$ and $n' \in N(M\hat{\mathbb{Z}})$, we have (identifying $n'$ with $1_{\infty} \times n'$)

$$
\phi_h(nn'g) = \phi_h(ng(g_{\text{fin}}^{-1}n'g_{\text{fin}})) = \phi_h(ng)
$$

For $n \in N(A)$ and $n' \in N(M\hat{\mathbb{Z}})$, we have (identifying $n'$ with $1_{\infty} \times n'$)
since by (12.17) $g_{\text{fin}}^{-1} n' g_{\text{fin}} = k^{-1} \delta n' \delta^{-1} k \in K_1(N)$. By strong approximation,

$$N(A) = N(Q)[N(R) \times N(M\hat{Z})],$$

so $N(Q) \setminus N(A) = N(M\hat{Z}) \setminus [N(R) \times N(M\hat{Z})]$. Note that the interval $[0, M]$ is a fundamental domain in $R \cong N(R)$ for $N(M\hat{Z}) \setminus N(R)$. Thus by the divorce theorem we have

$$W_\beta(g) = \int_{N(Q) \setminus N(A)} \phi_h(ng)\theta_\beta(n)dn$$

$$= \int_0^M \int_{N(M\hat{Z})} \phi_h\left(\begin{array}{cc} 1 & \beta \\ 0 & 1 \end{array}\right) n' g \theta_{\beta, \infty}(t) \theta_{\beta, \text{fin}}(n')dn' dt$$

$$= \int_0^M \phi_h\left(\begin{array}{cc} 1 & \beta \\ 0 & 1 \end{array}\right) g \theta_{\infty}(\beta t)dt \int_{M\hat{Z}} \theta_{\text{fin}}(\beta a)da.$$ Note that the integral over $M\hat{Z}$ is nonzero if and only if $\beta M\hat{Z} \subset \hat{Z}$, i.e. if and only if $\beta = \frac{n}{M}$. Then by (12.18), the above is

$$= \text{meas}(M\hat{Z}) \int_0^M j(g_\infty, i)^{-k} h_\delta(z + t)e^{-2\pi i n\omega} dt$$

$$= j(g_\infty, i)^{-k} \frac{1}{M} \int_0^M h_\delta(z + t)e^{-2\pi i n\omega/M} dt$$

$$= j(g_\infty, i)^{-k} e^{2\pi i n\omega/M} a_n(\delta)$$

by (3.20) on page 17. □

12.4. Characterizing the image of $S_k(N, \omega')$ in $L_0^2(\omega)$.

**Proposition 12.5.** Let $A_k(N, \omega)$ be the space of all functions $\varphi \in L_0^2(\omega)$ satisfying

(a) $\varphi(gk) = \omega(k)\varphi(g)$ for all $k \in K_0(N)$ and $g \in G(A)$

(b) $\varphi(g \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}) = e^{ik\theta} \varphi(g)$ for all $\theta$ and all $g \in G(A)$

(c) The function $\varphi$ satisfies

$$R(E)^{-1} \varphi = 0,$$

where we take $R(E)^{-1} \varphi(g) = \left. \frac{d}{dt} \right|_{t=0} R(\exp(tE^-) \times 1_{\text{fin}}) \varphi(g)$.

Then the map $h \mapsto \phi_h$ defines an isometry from $S_k(N, \omega')$ onto $A_k(N, \omega)$.

**Remarks:** (i) For any function $\varphi$ that transforms under $Z(A)$ by $\omega$, condition (a) is equivalent to:

$$\varphi(gk) = \varphi(g)$$

for all $k \in K_1(N)$.

(ii) Condition (c) can be replaced by

$$R(\Delta) \varphi = \frac{k}{2}(1 - \frac{k}{2}) \varphi.$$

This can be seen using Theorem 12.6 below and Theorem 11.44.
PROOF. Let \( h \in S_k(N, \omega') \). We begin by showing that \( \phi_h \) satisfies conditions (a), (b) and (c). Write \( g = \gamma(g_\infty \times k_0) \) for \( \gamma \in G(\mathbb{Q}), \ g_\infty \in G(\mathbb{R})^+, \) and \( k_0 \in K_0(N) \). For any \( k \in K_0(N) \),

\[
\phi_h(gk) = \phi_h(\gamma(g_\infty \times k_0)) = \omega(k)\omega(k_0)j(g_\infty, i)^{-k}h(g_\infty(i)) = \omega(k)\phi_h(g).
\]

Thus \( \phi_h \) satisfies condition (a).

Let \( k_\theta = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \). Then \( k_\theta \) stabilizes \( i \), so for \( g = \gamma(g_\infty \times k) \),

\[
\phi_h(gk_\theta) = \omega(k)j(g_\infty k_\theta, i)^{-k}h(g_\infty k_\theta(i)) = \omega(k)j(g_\infty, k_\theta(i))^{-k}j(k_\theta, i)^{-k}h(g_\infty(i)) = e^{ik\theta}\phi_h(g).
\]

This proves condition (b) for \( \phi_h \).

Let \( L \) denote the left regular representation \( L(g)f(x) = f(g^{-1}x) \). It is clear that \( L \) commutes with the right regular action of the Lie algebra, i.e. \( L(g)R(X) = R(X)L(g) \). Therefore for any \( g \in G(\mathbb{A}) \),

\[
R(E^-)\phi_h(g_\infty \times g_{\text{fin}}) = L(\varepsilon_\infty \times g_{\text{fin}})^{-1}R(E^-)\phi_h(\varepsilon_\infty g_\infty \times 1_{\text{fin}}),
\]

where \( \varepsilon_\infty = \left( \begin{array}{c} 1 \\ -1 \end{array} \right)^{\text{sgn(det } g_\infty)} \). Thus in order to verify condition (c) it suffices to show that \( R(E^-)\phi_h(g_\infty \times 1_{\text{fin}}) = 0 \) for \( g_\infty \in G(\mathbb{R})^+ \). Write

\[
g_\infty = z_\infty \left( \begin{array}{cc} 1 & x \\ 1 & y \end{array} \right) \left( \begin{array}{c} y^{1/2} \\ y^{-1/2} \end{array} \right) k_\theta \in G(\mathbb{R})^+.
\]

Note that

\[
\phi_h(g_\infty \times 1_{\text{fin}}) = y^{k/2}e^{ik\theta}h(x + iy).
\]

Recall from Proposition 11.37 that as an operator on \( C^\infty(G(\mathbb{R})^+) \),

\[
R(E^-) = e^{-2i\theta} \left( -2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + i\frac{\partial}{\partial \theta} \right).
\]

Let \( z = x + iy \). Then using subscripts to denote partial derivatives,

\[
R(E^-)\phi_h(g_\infty \times 1_{\text{fin}}) = R(E^-)y^{k/2}e^{ik\theta}h(z)
\]

\[
= e^{-2i\theta}(-2iy^{\frac{k}{2}+1}e^{ik\theta}h_x(z) + 2y\left( y^{\frac{k}{2}}e^{ik\theta}h_y(z) + \frac{k}{2}y^{\frac{k}{2}-1}e^{ik\theta}h(z) \right) - ke^{ik\theta}y^{k/2}h(z))
\]

\[
= e^{-2i\theta}\left( -2iy^{\frac{k}{2}+1}e^{ik\theta}h_x(z) + 2y^{\frac{k}{2}+1}e^{ik\theta}h_y(z) + ky^{k/2}e^{ik\theta}h(z) - ke^{ik\theta}y^{k/2}h(z) \right)
\]

\[
= -2ie^{(k-2)i\theta}y^{\frac{k}{2}+1}(h_x(z) + ih_y(z))
\]

(12.19)

\[
= -4ie^{(k-2)i\theta}y^{\frac{k}{2}+1}\frac{\partial h}{\partial \bar{z}}(z).
\]

Because \( h \) is holomorphic, the above is identically 0. This proves that \( \phi_h \) satisfies condition (c).
Let $D_N \subset \mathbf{H}$ be a fundamental domain for $\Gamma_0(N) \setminus \mathbf{H}$. We identify $D_N$ in the usual way with a subset of $\text{SL}_2(\mathbf{R})$ (cf. Proposition 7.43, p. 104). For the square-integrability of $\phi_h$, we use Proposition 7.43 to compute

$$
\int_{\mathcal{C}(Q) \setminus \mathcal{C}(A)} |\phi_h(g)|^2 dg = \int_{D_N K_\infty \times K_0(N)} |\phi_h(g)|^2 dg
$$

$$
= \int_{D_N K_\infty \times K_0(N)} |j(g_\infty, i)^{-k} h(g_\infty(i))|^2 dg
$$

\[(12.20) = \text{meas}(K_0(N)) \int \int_{D_N} |y^{k/2} h(x + iy)|^2 dx \, dy / y^2 \]

$$
= \frac{1}{\psi(N)} \int \int_{\Gamma_0(N) \setminus \mathbf{H}} |h(x + iy)|^2 y^{k} dx \, dy / y^2.
$$

Thus the $L^2$-norm of $\phi_h$ equals the Petersson norm of $h$. (We are using the fact that $\text{meas}(K_0(N)) = 1/\psi(N)$. See the beginning of Section 13.) It follows that $\phi_h$ is square-integrable, and because its constant term vanishes by (12.15) of Proposition 12.2, we see that $\phi_h \in L^2_0(\omega)$. Recall that the fact that $\phi_h(zg) = \omega(z) \phi_h(g)$ was shown in (12.13) above. This completes the proof that $\phi_h \in A_k(N, \omega)$.

Conversely, suppose $\varphi \in A_k(N, \omega)$. Define a function $h$ on the upper half plane in the following way. For $z \in \mathbf{H}$, choose $g_\infty \in G(\mathbf{R})^+$ such that $g_\infty(i) = z$, and let

$$
h(z) = h(g_\infty(i)) = j(g_\infty, i)^k \varphi(g_\infty \times 1_\text{fin}).
$$

Using the fact that the stabilizer of $i \in \mathbf{H}$ in $G(\mathbf{R})^+$ is $Z(\mathbf{R})K_\infty$, it is straightforward to check that $h(z)$ is independent of the choice of $g_\infty$. Using (12.12), define a function $\phi_h$ on $G(A)$. Then $\phi_h = \varphi$ since

$$
\phi_h(\gamma)(g_\infty \times k)) = \omega(k) j(g_\infty, i)^{-k} h(g_\infty(i)) = \omega(k) \varphi(g_\infty \times 1_\text{fin}) = \varphi(\gamma)(g_\infty \times k)).
$$

Let

$$
g_\infty = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} \\ y^{-1/2} \end{pmatrix}.
$$

Then $z = g_\infty(i) = x + iy$ and $h(x + iy) = y^{-k/2} \varphi(g_\infty \times 1_\text{fin})$. Using condition (c) and (12.19) with $\theta = 0$, we have

$$
\frac{\partial h}{\partial \bar{z}} = -\frac{1}{4iy^{k/2+1}} \text{Re}(E^-) \varphi(g_\infty \times 1_\text{fin}) = 0.
$$

Thus $h$ is holomorphic.

Write $z = g_\infty(i)$ as above, and let $\gamma \in \Gamma_0(N)$. Then

$$
h(\gamma z) = h(\gamma g_\infty(i)) = j(\gamma g_\infty, i)^k \varphi(\gamma g_\infty \times 1_\text{fin})
$$

$$
= j(\gamma, g_\infty(i))^k \varphi(g_\infty \times \gamma^{-1}_\text{fin})
$$

$$
= \omega'(\gamma)^{-1} j(\gamma, z)^k h(z).
$$

Thus $h$ is weakly modular.
To show that \( h \in S_k(N, \omega') \), it remains to check that \( h \) vanishes at the cusps of \( \Gamma_1(N) \). This is a consequence of the square-integrability of \( \phi \). By (12.20) we see that the Petersson norm of \( h \) is finite. By Proposition 3.39, \( h \) is a cusp form. Hence \( h \mapsto \phi_h \) is surjective. Because it is a norm-preserving linear map, it must also be injective, so the proposition is proven. \( \square \)

As mentioned earlier, the right regular representation of \( G(\mathbb{A}) \) on \( L^2_0(\omega) \) decomposes into an orthogonal Hilbert space direct sum of irreducible representations

\[ R_0 \cong \bigoplus \pi, \]

where \( \pi \) are (by definition) the cuspidal automorphic representations of \( G(\mathbb{A}) \) with central character \( \omega \).

Each cuspidal representation \( \pi \) is isomorphic to a restricted tensor product

\[ \pi \cong \bigotimes_{p < \infty} \pi_p \]

where \( \pi_p \) is an irreducible admissible representation of \( G(\mathbb{Q}_p) \). For the proof and a rigorous statement, see Section 3.3.3 of \([GGPS]\). A more general factorization theorem which applies to all irreducible admissible representations of \( G(\mathbb{A}) \) (\( G \) any reductive group) was given by Flath, \([Fl]\). See also Section 3.4 of \([Bu]\) for the case of \( \text{GL}_n(\mathbb{A}) \). For the present purpose, it is enough to know that \( \pi \) is a tensor product

\[ \pi = \pi_\infty \otimes \pi_{\text{fin}} \]

where \( \pi_\infty \) (resp. \( \pi_{\text{fin}} \)) is an irreducible unitary representation of \( G(\mathbb{R}) \) (resp. \( G(\mathbb{A}_{\text{fin}}) \)). The isomorphism class of \( \pi_\infty \) is called the infinity type of \( \pi \). When \( \pi_\infty \cong \pi_k \), we let \( v_\infty \in V_{\pi_\infty} \) denote a lowest weight vector (unique up to scalars). For any representation \( \pi_{\text{fin}} \) of \( G(\mathbb{A}_{\text{fin}}) \) and any subgroup \( U \) of \( G(\mathbb{A}_{\text{fin}}) \), let \( \pi_{\text{fin}}^U \) denote the subspace of \( U \)-fixed vectors in the space of \( \pi_{\text{fin}} \).

**Theorem 12.6.** With notation as above, we have

\[ A_k(N, \omega) = \bigoplus_{\pi_\infty \cong \pi_k} \mathbb{C}v_\infty \otimes \pi_{\text{fin}}^{K_1(N)} \]

where the sum taken is over all cuspidal representations in \( L^2_0(\omega) \) of the form \( \pi = \pi_k \otimes \pi_{\text{fin}} \).

**Proof.** Suppose \( \phi = v_\infty \otimes v_{\text{fin}} \) belongs to one of the summands on the right-hand side of (12.21). To show that \( \phi \in A_k(N, \omega) \), we check that \( \phi \) satisfies conditions \((a')\), \((b)\) and \((c)\) of Proposition 12.5. This is straightforward, using Theorem 11.44:

\((a')\) For \( k \in K_1(N) \),

\[ R(1_\infty \times k)\phi = \pi_\infty(1)v_\infty \otimes \pi_{\text{fin}}(k)v_{\text{fin}} = v_\infty \otimes v_{\text{fin}} = \phi. \]
(b) For \( k_\theta \in K_\infty \),
\[
R(k_\theta \times 1_{\text{fin}})\phi = \pi_\infty(k_\theta)v_\infty \otimes \pi_{\text{fin}}(1)v_{\text{fin}} = e^{ik_\theta}v_\infty \otimes v_{\text{fin}} = e^{ik_\theta} \phi
\]
since \( v_\infty \) is a lowest weight vector for \( \pi_\infty \cong \pi_k \) (cf. Theorem 11.44).

(c) Lastly,
\[
R(E^-)\phi = \left. \frac{d}{dt} \right|_{t=0} R(\exp(tE^-) \times 1_{\text{fin}})\phi
\]
\[
= \left. \frac{d}{dt} \right|_{t=0} \pi_\infty(\exp(tE^-))v_\infty \otimes \pi_{\text{fin}}(1)v_{\text{fin}}
\]
\[
= (\pi_\infty(E^-)v_\infty) \otimes v_{\text{fin}} = 0,
\]
again by Theorem 11.44.

Conversely, suppose \( \phi \in A_k(N, \omega) \) is nonzero. We need to show that \( \phi \) belongs to the right-hand side of (12.21). For any cuspidal \( \pi \), let \( V_\pi \) be the space of \( \pi \), so \( L_0^2(\omega) \) is the closure of \( \bigoplus V_\pi \). Let \( p_\pi : L_0^2(\omega) \to V_\pi \) be the orthogonal projection map. Then \( p_\pi \) intertwines the action of \( R \) since \( V_\pi \) is a closed stable subspace and \( R \) is unitary. Using this fact, it is straightforward to show that \( p_\pi(A_k(N, \omega)) \subset A_k(N, \omega) \), and hence
\[
p_\pi(A_k(N, \omega)) = A_k(N, \omega) \cap V_\pi.
\]
It follows that \( A_k(N, \omega) \) is the closure of \( \bigoplus_{\pi}(V_\pi \cap A_k(N, \omega)) \). However this direct sum is finite-dimensional, hence already closed, so
\[
A_k(N, \omega) = \bigoplus_{\pi}(V_\pi \cap A_k(N, \omega)).
\]

By this fact, it suffices to consider the case where
\[
\phi \in V_\pi \cap A_k(N, \omega)
\]
for some cuspidal \( \pi \).

It remains to show that \( \pi_\infty \cong \pi_k \) and \( \phi \in \mathbb{C}v_\infty \otimes \pi_{\text{fin}}^{K_1(N)} \). By linearity, we can assume that \( \phi = v_\infty \otimes v_{\text{fin}} \) for some nonzero \( v_\infty \in V_{\pi_\infty} \) and \( v_{\text{fin}} \in V_{\text{fin}} \).

Let
\[
V_\infty(k) = \{ v \in V_{\pi_\infty} | \pi_\infty(k_\theta)v = e^{ik_\theta}v \}
\]
be the isotypic component in \( V_{\pi_\infty} \) of the character \( k_\theta \mapsto e^{ik_\theta} \) of \( K_\infty \). Note that by property (b),
\[
\pi_\infty(k_\theta)v_\infty \otimes v_{\text{fin}} = \pi(k_\theta \times 1_{\text{fin}})\phi
\]
\[
= e^{ik_\theta} \phi = e^{ik_\theta}v_\infty \otimes v_{\text{fin}}.
\]
This proves that \( v_\infty \in V_\infty(k) \). By a similar argument, we see easily that \( v_{\text{fin}} \in \pi_{\text{fin}}^{K_1(N)} \), and hence \( \phi \in V_\infty(k) \otimes \pi_{\text{fin}}^{K_1(N)} \).

Now because \( \phi \) satisfies condition (c),
\[
0 = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(tE^-) \times 1_{\text{fin}})\phi = \left. \frac{d}{dt} \right|_{t=0} \pi_\infty(\exp(tE^-))v_\infty \otimes v_{\text{fin}}
\]
\[
= (\pi_\infty(E^-)v_\infty) \otimes v_{\text{fin}}.
\]
Thus \( \pi_\infty(E^{-})v_\infty = 0 \). We have now shown that \( v_\infty \) satisfies (2) of Theorem 11.44, and hence \( \pi_\infty \cong \pi_k \) and \( v_\infty \) is a lowest weight vector. \( \square \)

Remark: If \( h \in S_k(N, \omega') \) is a Hecke eigenform, the cuspidal representation \( \pi \) generated by \( \phi_h \in A_k(N, \omega) \subset L^2_0(\omega) \) is irreducible and has \( \pi_\infty = \pi_k \). For details about the correspondence (1-1 only at the level of newforms) between \( h \) and \( \pi \), including a description of the local factors \( \pi_p \), see [G1] or [Ro1].

13. Construction of the test function \( f \)

We now construct a continuous function \( f \in L^1(G(A), \omega^{-1}) \) such that the trace of \( R(f) \) on \( L^2(\omega) \) gives the trace of the Hecke operator \( T_n \) on \( S_k(N, \omega') \). The function \( f \) will be a product of local functions on \( G(Q_p) \), i.e.

\[
 f = f_\infty \times f_n, \quad \text{where} \quad f_n = \prod_{p<\infty} f_n^p.
\]

13.1. The non-archimedean component of \( f \). The idea is to define \( f_n^p \) using double cosets as in the construction of \( T_n \), using \( K_0(N) \) in place of \( \Gamma_0(N) \).

**Lemma 13.1.** Suppose \( p|N \). Then

\[
 K_p = \bigcup_{\delta \in \mathbb{Z}_p/N\mathbb{Z}_p} \left( \begin{array}{cc} \delta & 1 \\ 1 & 0 \end{array} \right) K_0(N)_p \cup \bigcup_{\tau \in p\mathbb{Z}_p/N\mathbb{Z}_p} \left( \begin{array}{cc} 1 & \tau \\ \tau & 1 \end{array} \right) K_0(N)_p,
\]

a disjoint union.

**Proof.** Let \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_p \). If \( p|c \), then \( a \in \mathbb{Z}_p^* \), so \( \left( \begin{array}{cc} a^{-1} & -b \\ 0 & ad-bc \end{array} \right) \in K_0(N)_p \), and

\[
 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} a^{-1} & -b \\ 0 & ad-bc \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ c/a & 1 \end{array} \right).
\]

Because we can further multiply by \( \left( \begin{array}{cc} 1 & 0 \\ N & 1 \end{array} \right) \) to obtain \( \left( \begin{array}{cc} 1 & 0 \\ c/a + N & 1 \end{array} \right) \), the entry \( \tau = c/a \in p\mathbb{Z}_p \) is unique modulo \( N\mathbb{Z}_p \).

If \( c \) is a unit, then multiplying by \( \left( \begin{array}{cc} c^{-1} & d \\ 0 & ad-c \end{array} \right) \) gives \( \left( \begin{array}{cc} a/c & 1 \\ 1 & 0 \end{array} \right) \). Once again, \( \delta = a/c \in \mathbb{Z}_p \) is unique modulo \( N\mathbb{Z}_p \).

This proves the decomposition. To see that it is disjoint, note that

\[
 \left( \begin{array}{cc} 1 & 0 \\ \tau & 1 \end{array} \right) \left( \begin{array}{cc} w & x \\ Ny & z \end{array} \right) = \left( \begin{array}{cc} * & * \\ \tau w + Ny & * \end{array} \right),
\]

which cannot equal \( \left( \begin{array}{cc} \delta & 1 \\ 1 & 0 \end{array} \right) \) since \( p|N+\tau \).

Define

\[
 \psi_p(N) = [K_p : K_0(N)_p].
\]