Contents

Preface vii
Introduction ix

Chapter 0. Preliminaries 1
  0.1 Notations 1
  0.2 Positive Semidefinite Matrices 1

Chapter 1. Positive Polynomials and Sums of Squares 3
  1.1 Preliminaries on Polynomials 3
  1.2 Positive Polynomials 4
  1.3 Extending Positive Polynomials 8
  1.4 Hilbert’s 17th Problem 11
  1.5 Baer-Krull Theorem 14
  1.6 Formal Power Series Rings 17

Chapter 2. Krivine’s Positivstellensatz 21
  2.1 Quadratic Modules and Preorderings 21
  2.2 Positivstellensatz 25
  2.3 The Proof 27
  2.4 The Real Spectrum 29
  2.5 Abstract Positivstellensatz 31
  2.6 Saturation 33
  2.7 Low-Dimensional Examples 35

Chapter 3. The Moment Problem 41
  3.1 Introduction 41
  3.2 Proof of Haviland’s Theorem 44
  3.3 Uniqueness Question 46
  3.4 The Conditions (SMP) and (MP) 47
  3.5 Schmüdgen’s Theorem 48
  3.6 Countable Dimensional Vector Spaces 50

Chapter 4. Non-Compact Case 55
  4.1 Stability 55
  4.2 Examples where (SMP) and (MP) fail 61
  4.3 Examples where (SMP) and (MP) hold 64
  4.4 Direct Integral Decomposition 65

Chapter 5. Archimedean T-modules 71
  5.1 Preprimes 71
5.2 $T$-modules 72
5.3 Semiorderings and Valuations 75
5.4 Representation Theorem 78
5.5 Theorems of Pólya and Reznick 80
5.6 Other Applications 83
5.7 Topology on $V_A = \text{Hom}(A, \mathbb{R})$ 84

Chapter 6. Schmüdgen’s Positivstellensatz 87
6.1 Wörmann’s Trick 87
6.2 Non-Compact Case 89
6.3 Remarks and Examples 92

Chapter 7. Putinar’s Question 97
7.1 Introduction 97
7.2 Stable Compactness 100
7.3 Jacobi-Prestel Counterexample 103
7.4 The Case $\dim \frac{\mathbb{R}[X]}{M-\sqrt{M}} \leq 1$ 105

Chapter 8. Weak Isotropy of Quadratic Forms 109
8.1 Isotropy and Weak Isotropy 109
8.2 Residue Forms 110
8.3 Local-Global Principle for Weak Isotropy 113
8.4 Pfister Forms 116
8.5 Application to Putinar’s Question 117

Chapter 9. Scheiderer’s Local-Global Principle 123
9.1 Basic Lemma 123
9.2 Local-Global Principle 125
9.3 The Case $n = 1$ 128
9.4 The Case $n = 2$ 130
9.5 Hessian Conditions 133
9.6 Second Local-Global Principle 134

Chapter 10. Semidefinite Programming and Optimization 137
10.1 The Cone of PSD Matrices 137
10.2 Semidefinite Programming 138
10.3 Max-Cut Problem 142
10.4 Global Optimization 145
10.5 Constrained Optimization 148
10.6 Exploiting the Gradient Ideal 151
10.7 Existence of Feasible Solutions 156

Appendix 1. Tarski-Seidenberg Theorem 161
11.1 Basic Version 161
11.2 Tarski’s Transfer Principle 162
11.3 Lang’s Homomorphism Theorem 163
11.4 Geometric Version 165
11.5 General Version 167

Appendix 2. Algebraic Sets 169
12.1 Transcendence Degree and Krull Dimension 169
CONTENTS

12.2 Non-Singular Zeros 171
12.3 Algebraic Sets 173
12.4 Dimension 175
12.5 Radical Ideals and Real Ideals 177
12.6 Simple Point Criterion 178
12.7 Sign-Changing Criterion 178

Bibliography 183