CHAPTER 1

Generalized Argument Principle and Rouché’s Theorem

In this chapter we review the results of Gohberg and Sigal in [114] concerning the generalization to operator-valued functions of two classical results in complex analysis, the *argument principle* and *Rouché’s theorem*.

To state the argument principle, we first observe that if \( f \) is holomorphic and has a zero of order \( n \) at \( z_0 \), we can write \( f(z) = (z-z_0)^n g(z) \), where \( g \) is holomorphic and nowhere vanishing in a neighborhood of \( z_0 \), and therefore

\[
\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.
\]

Then the function \( f'/f \) has a simple pole with residue \( n \) at \( z_0 \). A similar fact also holds if \( f \) has a pole of order \( n \) at \( z_0 \), that is, if \( f(z) = (z-z_0)^{-n} h(z) \), where \( h \) is holomorphic and nowhere vanishing in a neighborhood of \( z_0 \). Then

\[
\frac{f'(z)}{f(z)} = -\frac{n}{z-z_0} + \frac{h'(z)}{h(z)}.
\]

Therefore, if \( f \) is holomorphic, the function \( f'/f \) will have simple poles at the zeros and poles of \( f \), and the residue is simply the order of the zero of \( f \) or the negative of the order of the pole of \( f \).

The argument principle results from an application of the residue formula. It asserts the following.

**Theorem 1.1 (Argument principle).** Let \( V \subset \mathbb{C} \) be a bounded domain with smooth boundary \( \partial V \) positively oriented and let \( f(z) \) be a meromorphic function in a neighborhood of \( \overline{V} \). Let \( P \) and \( N \) be the number of poles and zeros of \( f \) in \( V \), counted with their multiplicities. If \( f \) has no poles and never vanishes on \( \partial V \), then

\[
\frac{1}{2\pi \sqrt{-1}} \int_{\partial V} \frac{f'(z)}{f(z)} \, dz = N - P.
\]

Rouché’s theorem is a consequence of the argument principle [237]. It is in some sense a continuity statement. It says that a holomorphic function can be perturbed slightly without changing the number of its zeros. It reads as follows.

**Theorem 1.2 (Rouché’s theorem).** With \( V \) as above, suppose that \( f(z) \) and \( g(z) \) are holomorphic in a neighborhood of \( \overline{V} \). If \( |f(z)| > |g(z)| \) for all \( z \in \partial V \), then \( f \) and \( f + g \) have the same number of zeros in \( V \).

In order to explain the main results of Gohberg and Sigal in [114], we begin with the finite-dimensional case which was first considered by Keldyš in [152]; see also [183]. We proceed to generalize formula (1.1) in this case as follows. If a
matrix-valued function \( A(z) \) is holomorphic in a neighborhood of \( \overline{V} \) and is invertible in \( V \) except possibly at a point \( z_0 \in V \), then by Gaussian eliminations we can write

\[
A(z) = E(z)D(z)F(z) \quad \text{in } V,
\]

where \( E(z), F(z) \) are holomorphic and invertible in \( V \) and \( D(z) \) is given by

\[
D(z) = \begin{pmatrix}
(z - z_0)^{k_1} & 0 \\
0 & \cdots \\
0 & (z - z_0)^{k_n}
\end{pmatrix}.
\]

Moreover, the powers \( k_1, k_2, \ldots, k_n \) are uniquely determined up to a permutation.

Let \( \text{tr} \) denote the trace. By virtue of the factorization (1.2), it is easy to produce the following identity:

\[
\frac{1}{2\pi\sqrt{-1}} \text{tr} \int_{\partial V} A(z)^{-1} \frac{d}{dz} A(z) \, dz = \frac{1}{2\pi\sqrt{-1}} \text{tr} \int_{\partial V} \left( E(z)^{-1} \frac{d}{dz} E(z) + D(z)^{-1} \frac{d}{dz} D(z) + F(z)^{-1} \frac{d}{dz} F(z) \right) \, dz
\]

\[
= \frac{1}{2\pi\sqrt{-1}} \text{tr} \int_{\partial V} D(z)^{-1} \frac{d}{dz} D(z) \, dz = \sum_{j=1}^{n} k_j,
\]

which generalizes (1.1).

In the next sections, we will extend the above identity as well as the factorization (1.2) to infinite-dimensional spaces under some natural conditions.

1.1. Definitions and Preliminaries

In this section we introduce the notation we will use in the text, gather a few definitions, and present some basic results, which are useful for the statement of the generalized Rouché theorem.

1.1.1. Compact Operators. If \( B \) and \( B' \) are two Banach spaces, we denote by \( \mathcal{L}(B, B') \) the space of bounded linear operators from \( B \) into \( B' \). An operator \( K \in \mathcal{L}(B, B') \) is said to be compact provided \( K \) takes any bounded subset of \( B \) to a relatively compact subset of \( B' \), that is, a set with compact closure.

The operator \( K \) is said to be of finite rank if \( \text{Im}(K) \), the range of \( K \), is finite-dimensional. Clearly every operator of finite rank is compact.

The next result is called the Fredholm alternative. See, for example, [164].

**Proposition 1.3** (Fredholm alternative). Let \( K \) be a compact operator on the Banach space \( B \). For \( \lambda \in \mathbb{C}, \lambda \neq 0 \), \( (\lambda I - K) \) is surjective if and only if it is injective.

1.1.2. Fredholm Operators. An operator \( A \in \mathcal{L}(B, B') \) is said to be Fredholm provided the subspace \( \text{Ker} A \) is finite-dimensional and the subspace \( \text{Im} A \) is closed in \( B' \) and of finite codimension. Let \( \text{Fred}(B, B') \) denote the collection of all Fredholm operators from \( B \) into \( B' \). We can show that \( \text{Fred}(B, B') \) is open in \( \mathcal{L}(B, B') \).
Next, we define the index of \( A \in \text{Fred}(\mathcal{B}, \mathcal{B}') \) to be
\[
\text{ind } A = \dim \text{Ker } A - \text{codim } \text{Im } A.
\]
In finite dimensions, the index depends only on the spaces and not on the operator.

The following proposition shows that the index is stable under compact perturbations [164].

**Proposition 1.4.** If \( A : \mathcal{B} \to \mathcal{B}' \) is Fredholm and \( K : \mathcal{B} \to \mathcal{B}' \) is compact, then their sum \( A + K \) is Fredholm, and
\[
\text{ind } (A + K) = \text{ind } A.
\]
Proposition 1.4 is a consequence of the following fundamental result about the index of Fredholm operators.

**Proposition 1.5.** The mapping \( A \mapsto \text{ind } A \) is continuous in \( \text{Fred}(\mathcal{B}, \mathcal{B}') \); i.e., \( \text{ind} \) is constant on each connected component of \( \text{Fred}(\mathcal{B}, \mathcal{B}') \).

### 1.1.3. Characteristic Value and Multiplicity

We now introduce the notions of characteristic values and root functions of analytic operator-valued functions, with which the readers might not be familiar. We refer, for instance, to the book by Markus [175] for the details.

Let \( \mathcal{U}(z_0) \) be the set of all operator-valued functions with values in \( \mathcal{L}(\mathcal{B}, \mathcal{B}') \) which are holomorphic in some neighborhood of \( z_0 \), except possibly at \( z_0 \).

The point \( z_0 \) is called a **characteristic value** of \( A(z) \in \mathcal{U}(z_0) \) if there exists a vector-valued function \( \phi(z) \) with values in \( \mathcal{B} \) such that

(i) \( \phi(z) \) is holomorphic at \( z_0 \) and \( \phi(z_0) \neq 0 \),

(ii) \( A(z)\phi(z) \) is holomorphic at \( z_0 \) and vanishes at this point.

Here, \( \phi(z) \) is called a **root function** of \( A(z) \) associated with the characteristic value \( z_0 \). The vector \( \phi_0 = \phi(z_0) \) is called an **eigenvector**. The closure of the linear set of eigenvectors corresponding to \( z_0 \) is denoted by \( \text{Ker } A(z_0) \).

Suppose that \( z_0 \) is a characteristic value of the function \( A(z) \) and \( \phi(z) \) is an associated root function. Then there exists a number \( m(\phi) \geq 1 \) and a vector-valued function \( \psi(z) \) with values in \( \mathcal{B}' \), holomorphic at \( z_0 \), such that
\[
A(z)\phi(z) = (z - z_0)^m(\phi)\psi(z), \quad \psi(z_0) \neq 0.
\]

The number \( m(\phi) \) is called the **multiplicity** of the root function \( \phi(z) \).

For \( \phi_0 \in \text{Ker } A(z_0) \), we define the rank of \( \phi_0 \), denoted by \( \text{rank}(\phi_0) \), to be the maximum of the multiplicities of all root functions \( \phi(z) \) with \( \phi(z_0) = \phi_0 \).

Suppose that \( n = \dim \text{Ker } A(z_0) < +\infty \) and that the ranks of all vectors in \( \text{Ker } A(z_0) \) are finite. A system of eigenvectors \( \phi_0^j, j = 1, \ldots, n \), is called a **canonical system of eigenvectors** of \( A(z) \) associated to \( z_0 \) if their ranks possess the following property: for \( j = 1, \ldots, n \), \( \text{rank}(\phi_0^j) \) is the maximum of the ranks of all eigenvectors in the direct complement in \( \text{Ker } A(z_0) \) of the linear span of the vectors \( \phi_0^1, \ldots, \phi_0^{j-1} \).

We call
\[
N(A(z_0)) := \sum_{j=1}^n \text{rank}(\phi_0^j)
\]
the **null multiplicity** of the characteristic value \( z_0 \) of \( A(z) \). If \( z_0 \) is not a characteristic value of \( A(z) \), we put \( N(A(z_0)) = 0 \).
Suppose that $A^{-1}(z)$ exists and is holomorphic in some neighborhood of $z_0$, except possibly at $z_0$. Then the number
\[ M(A(z_0)) = N(A(z_0)) - N(A^{-1}(z_0)) \]
is called the multiplicity of $z_0$. If $z_0$ is a characteristic value and not a pole of $A(z)$, then $M(A(z_0)) = N(A(z_0))$ while $M(A(z_0)) = -N(A^{-1}(z_0))$ if $z_0$ is a pole and not a characteristic value of $A(z)$.

1.1.4. Normal Points. Suppose that $z_0$ is a pole of the operator-valued function $A(z)$ and the Laurent series expansion of $A(z)$ at $z_0$ is given by
\[ A(z) = \sum_{j \geq -s} (z - z_0)^j A_j. \]
(1.3)
If in (1.3) the operators $A_{-j}, j = 1, \ldots, s$, have finite-dimensional ranges, then $A(z)$ is called finitely meromorphic at $z_0$.

The operator-valued function $A(z)$ is said to be of Fredholm type (of index zero) at the point $z_0$ if the operator $A_0$ in (1.3) is Fredholm (of index zero).

If $A(z)$ is holomorphic and invertible at $z_0$, then $z_0$ is called a regular point of $A(z)$. The point $z_0$ is called a normal point of $A(z)$ if $A(z)$ is finitely meromorphic, of Fredholm type at $z_0$, and regular in a neighborhood of $z_0$ except at $z_0$ itself.

1.1.5. Trace. Let $A$ be a finite-dimensional operator acting from $\mathcal{B}$ into itself. There exists a finite-dimensional invariant subspace $\mathcal{C}$ of $A$ such that $A$ annihilates some direct complement of $\mathcal{C}$ in $\mathcal{B}$. We define the trace of $A$ to be that of $A|_{\mathcal{C}}$, which is given in the usual way. It is desirable to recall some results about the trace operator.

**Proposition 1.6.** The following results hold:

(i) $\text{tr} A$ is independent of the choice of $\mathcal{C}$, so that it is well-defined.

(ii) $\text{tr}$ is linear.

(iii) If $B$ is a finite-dimensional operator from $\mathcal{B}$ to itself, then
\[ \text{tr} AB = \text{tr} BA. \]

(iv) If $M$ is a finite-dimensional operator from $\mathcal{B} \times \mathcal{B}'$ to itself, given by
\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]
then $\text{tr} M = \text{tr} A + \text{tr} D$.

Recall that if an operator-valued function $C(z)$ is finitely meromorphic in the neighborhood $V$ of $z_0$, which contains no poles of $C(z)$ except possibly $z_0$, then $\int_{\partial V} C(z) \, dz$ is a finite-dimensional operator. The following identity will also be used frequently.

**Proposition 1.7.** Let $A(z)$ and $B(z)$ be two operator-valued functions which are finitely meromorphic in the neighborhood $\nabla$ of $z_0$, which contains no poles of $A(z)$ and $B(z)$ other than $z_0$. Then we have
\[ \text{tr} \int_{\partial V} A(z)B(z) \, dz = \text{tr} \int_{\partial V} B(z)A(z) \, dz. \]
1.2. Factorization of Operators

We say that $A(z) \in \mathcal{U}(z_0)$ admits a factorization at $z_0$ if $A(z)$ can be written as

\begin{equation}
A(z) = E(z)D(z)F(z),
\end{equation}

where $E(z), F(z)$ are regular at $z_0$ and

\begin{equation}
D(z) = P_0 + \sum_{j=1}^{n} (z - z_0)^k_j P_j.
\end{equation}

Here, $P_j$'s are mutually disjoint projections, $P_1, \ldots, P_n$ are one-dimensional operators, and $I - \sum P_j$ is a finite-dimensional operator.

**Theorem 1.8.** $A(z) \in \mathcal{U}(z_0)$ admits a factorization at $z_0$ if and only if $A(z)$ is finitely meromorphic and of Fredholm type of index zero at $z_0$.

**Proof.** Suppose that $A(z)$ is finitely meromorphic and of Fredholm type of index zero at $z_0$. We shall construct $E, F,$ and $D$ such that (1.5) holds. Write the Laurent series expansion of $A(z)$ as follows:

\[ A(z) = \sum_{j=-\nu}^{+\infty} (z-z_0)^j A_j \]

in some neighborhood $U$ of $z_0$. Since $\text{ind} A_0 = 0$, then by the Fredholm alternative $B_0 := A_0 + K_0$ is invertible for some finite-dimensional operator $K_0$. Consequently, $B(z) := K_0 + \sum_{j=0}^{+\infty} (z-z_0)^j A_j$ is invertible in some neighborhood $U_1$ of $z_0$ and

\begin{equation}
A(z) = C(z) + B(z) = B(z)[I + B^{-1}(z)C(z)],
\end{equation}

where

\[ C(z) = \sum_{j=-\nu}^{-1} (z-z_0)^j A_j - K_0. \]

Since $K(z) := B^{-1}(z)C(z)$ is finitely meromorphic, we can write $K(z)$ in the form

\[ K(z) = \sum_{j=1}^{\nu} (z-z_0)^{-j} K_j + T_1(z), \]

where $K_j, j = 1, \ldots, \nu,$ are finite-dimensional and $T_1$ is holomorphic.

Since the operators $A_j$ and $K_j$ are finite-dimensional, there exists a subspace $\mathcal{M}$ of $\mathcal{B}$ of finite codimension such that

\[
\begin{align*}
\mathcal{M} & \subset \ker A_j, \ j = -\nu, \ldots, -1, \\
\mathcal{M} & \subset \ker K_j, \ j = 0, \ldots, \nu, \\
\mathcal{M} \cap \text{im} K_j & = \{0\}, \ j = 1, \ldots, \nu.
\end{align*}
\]
Let $C$ be a direct finite-dimensional complement of $\mathfrak{N}$ in $\mathcal{B}$ and let $P$ be the projection onto $C$ satisfying $P(I - P) = 0$. Set $P_0 := I - P$. We have
\[ I + K(z) = I + PK(z)P + P_0K(z)P = I + PK(z)P + P_0T_1(z)P, \]
and therefore,
\[ (1.8) \quad I + K(z) = (I + PK(z)P)(I + P_0T_1(z)P). \]
Since $P(I + K(z))P$ can be viewed as an operator from $C$ into itself and $C$ is finite-dimensional, it follows from Gaussian elimination that
\[ P(I + K(z))P = E_1(z)D_1(z)F_1(z), \]
where $D_1(z)$ is diagonal and $E_1(z)$ and $F_1(z)$ are holomorphic and invertible. In view of (1.8), this implies that
\[
A(z) = B(z)(P_0 + P(I + K(z))P)(I + P_0T_1(z)P)
= B(z)(P_0 + E_1(z)D_1(z)F_1(z))(I + P_0T_1(z)P)
= B(z)(P_0 + E_1(z))(P_0 + D_1(z))(P_0 + F_1(z))(I + P_0T_1(z)P).
\]
Here $I + P_0T_1(z)P$ is holomorphic and invertible with inverse $I - P_0T_1(z)P$. Thus, taking
\[
E(z) := B(z)(P_0 + E_1(z)), \quad F(z) := (P_0 + F_1(z))(I + P_0T_1(z)P)
\]
yields the desired factorization for $A$ since $E(z)$ and $F(z)$, given by the above formulas, are holomorphic and invertible at $z_0$.

The converse result, that $A(z) = E(z)D(z)F(z)$ with $E(z), F(z)$ regular at $z_0$ and $D(z)$ satisfying (1.6) is finitely meromorphic and of Fredholm type of index zero at $z_0$, is easy. \(\square\)

**Corollary 1.9.** $A(z)$ is normal at $z_0$ if and only if $A(z)$ admits a factorization such that $I = \sum_{j=0}^{n} P_j$ in (1.6). Moreover, we have
\[ M(A(z_0)) = k_1 + \cdots + k_n \]
for $k_1, \cdots, k_n$, given by (1.6).

**Corollary 1.10.** Every normal point of $A(z)$ is a normal point of $A^{-1}(z)$.

### 1.3. Main Results of the Gohberg and Sigal Theory

We now tackle our main goal of this chapter, which is to generalize the argument principle and Rouché’s theorem to operator-valued functions.

#### 1.3.1. Argument Principle

Let $V$ be a simply connected bounded domain with rectifiable boundary $\partial V$. An operator-valued function $A(z)$ which is finitely meromorphic and of Fredholm type in $V$ and continuous on $\partial V$ is called *normal* with respect to $\partial V$ if the operator $A(z)$ is invertible in $\overline{V}$, except for a finite number of points of $V$ which are normal points of $A(z)$.

**Lemma 1.11.** An operator-valued function $A(z)$ is normal with respect to $\partial V$ if it is finitely meromorphic and of Fredholm type in $V$, continuous on $\partial V$, and invertible for all $z \in \partial V$.
PROOF. To prove that $A$ is normal with respect to $\partial V$, it suffices to prove that $A(z)$ is invertible except at a finite number of points in $V$. To this end choose a connected open set $U$ with $\overline{U} \subset V$ so that $A(z)$ is invertible in $V \setminus U$. Then, for each $\xi \in U$, there exists a neighborhood $U_\xi$ of $\xi$ in which the factorization (1.5) holds. In $U_\xi$, the kernel of $A(z)$ has a constant dimension except at $\xi$. Since $\overline{U}$ is compact, we can find a finite covering of $\overline{U}$, i.e.,

$$\overline{U} \subset U_{\xi_1} \cup \cdots \cup U_{\xi_k},$$

for some points $\xi_1, \ldots, \xi_k \in U$. Therefore, $\dim \ker A(z)$ is constant in $V \setminus \{\xi_1, \ldots, \xi_k\}$, and so $A(z)$ is invertible in $V \setminus \{\xi_1, \ldots, \xi_k\}$. \hfill $\square$

Now, if $A(z)$ is normal with respect to the contour $\partial V$ and $z_i, i = 1, \ldots, \sigma$, are all its characteristic values and poles lying in $V$, we put

$$M(A(z); \partial V) = \sum_{i=1}^{\sigma} M(A(z_i)).$$

(1.9)

The full multiplicity $M(A(z); \partial V)$ of $A(z)$ in $V$ is the number of characteristic values of $A(z)$ in $V$, counted with their multiplicities, minus the number of poles of $A(z)$ in $V$, counted with their multiplicities.

THEOREM 1.12 (Generalized argument principle). Suppose that the operator-valued function $A(z)$ is normal with respect to $\partial V$. Then we have

$$\mathcal{M}(A(z); \partial V) = \frac{1}{2\sqrt{-1}\pi} \tr \int_{\partial V} A^{-1}(z) \frac{d}{dz} A(z) dz.$$  

(1.10)

PROOF. Let $z_j, j = 1, \ldots, \sigma$, denote all the characteristic values and all the poles of $A$ lying in $V$. The key of the proof lies in using the factorization (1.5) in each of the neighborhoods of the points $z_j$. We have

$$\frac{1}{2\sqrt{-1}\pi} \tr \int_{\partial V} A^{-1}(z) \frac{d}{dz} A(z) dz = \sum_{j=1}^{\sigma} \frac{1}{2\sqrt{-1}\pi} \tr \int_{\partial V_j} A^{-1}(z) \frac{d}{dz} A(z) dz,$$

(1.11)

where, for each $j$, $V_j$ is a neighborhood of $z_j$. Moreover, in each $V_j$, the following factorization of $A$ holds:

$$A(z) = E^{(j)}(z)D^{(j)}(z)F^{(j)}(z), \quad D^{(j)}(z) = P_0^{(j)} + \sum_{i=1}^{n_j} (z - z_j)^{k_{ij}} P_i^{(j)}.$$

As for the matrix-valued case at the beginning of this chapter, it is readily verified that

$$\frac{1}{2\sqrt{-1}\pi} \tr \int_{\partial V_j} A^{-1}(z) \frac{d}{dz} A(z) dz = \frac{1}{2\sqrt{-1}\pi} \tr \int_{\partial V_j} (D^{(j)}(z))^{-1} \frac{d}{dz} D^{(j)}(z) dz$$

$$= \sum_{i=1}^{n_j} k_{ij} = M(A(z_j)).$$

Now, (1.10) follows by using (1.11). \hfill $\square$

The following is an immediate consequence of Lemma 1.11, identity (1.10), and (1.4).
Corollary 1.13. If the operator-valued functions \( A(z) \) and \( B(z) \) are normal with respect to \( \partial V \), then \( C(z) := A(z)B(z) \) is also normal with respect to \( \partial V \), and

\[
\mathcal{M}(C(z); \partial V) = \mathcal{M}(A(z); \partial V) + \mathcal{M}(B(z); \partial V).
\]

The following general form of the argument principle will be useful. It can be proven by the same argument as the one in Theorem 1.12.

Theorem 1.14. Suppose that \( A(z) \) is an operator-valued function which is normal with respect to \( \partial V \). Let \( f(z) \) be a scalar function which is analytic in \( V \) and continuous in \( \overline{V} \). Then

\[
\frac{1}{2\sqrt{-1}\pi} \text{tr} \int_{\partial V} f(z)A^{-1}(z) \frac{d}{dz} A(z)dz = \sum_{j=1}^{\sigma} M(A(z_j))f(z_j),
\]

where \( z_j, j = 1, \ldots, \sigma \), are all the points in \( V \) which are either poles or characteristic values of \( A(z) \).

1.3.2. Generalization of Rouché’s Theorem. A generalization of Rouché’s theorem to operator-valued functions is stated below.

Theorem 1.15 (Generalized Rouché’s theorem). Let \( A(z) \) be an operator-valued function which is normal with respect to \( \partial V \). If an operator-valued function \( S(z) \) which is finitely meromorphic in \( V \) and continuous on \( \partial V \) satisfies the condition

\[
\|A^{-1}(z)S(z)\|_{\mathcal{L}(\mathcal{B}, \mathcal{B})} < 1, \quad z \in \partial V,
\]

then \( A(z) + S(z) \) is also normal with respect to \( \partial V \) and

\[
\mathcal{M}(A(z); \partial V) = \mathcal{M}(A(z) + S(z); \partial V).
\]

Proof. Let \( C(z) := A^{-1}(z)S(z) \). By Corollary 1.10, \( C(z) \) is finitely meromorphic in \( V \). Suppose that \( z_1, z_2, \ldots, z_n \), are all of the poles of \( C(z) \) in \( V \) and that \( C(z) \) has the following Laurent series expansion in some neighborhood of each \( z_j \):

\[
C(z) = \sum_{k=-\infty}^{+\infty} (z - z_j)^k C_k^{(j)}.
\]

Let \( \mathcal{H} \) be the intersection of the kernels \( \text{Ker} C_k^{(j)} \) for \( j = 1, \ldots, n \) and \( k = 1, \ldots, \nu_j \). Then, \( \dim \mathcal{B}/\mathcal{H} < +\infty \) and the restriction \( C(z)|_{\mathcal{H}} \) of \( C(z) \) to \( \mathcal{H} \) is holomorphic in \( V \).

Let \( q := \max_{z \in \partial V} \|C(z)\| \), which by assumption is less than 1. Since

\[
\Delta_z \|C(z)|_{\mathcal{H}}\|^2 = 4\| \frac{\partial}{\partial z} C(z)|_{\mathcal{H}} \|^2,
\]

then \( \|C(z)|_{\mathcal{H}}\| \) is subharmonic in \( V \), and hence we have from the maximum principle

\[
\max_{z \in V} \|C(z)|_{\mathcal{H}}\| \leq q.
\]

It then follows that

\[
\|(I + C(z))x\| \geq (1 - q)\|x\|, \quad x \in \mathcal{H}, z \in V.
\]
This implies that \((I + C(z))|_{\partial V}\) has a closed range and \(\text{Ker}(I + C(z))|_{\partial V} = 0\). Therefore, \(I + C(z)\) has a closed range and a kernel of finite dimension for \(z \in V \setminus \{z_1, \ldots, z_n\}\). By a slight extension of Proposition 1.5 \cite{241}, \(I(z)\) defined by

\[
I(z) = \dim \text{Ker}(I + C(z)) - \text{codim} \text{Im}(I + C(z))
\]

is continuous for \(z \in V \setminus \{z_1, \ldots, z_n\}\). Thus,

\[
\text{ind}(I + C(z)) = 0 \quad \text{for} \quad z \in V \setminus \{z_1, \ldots, z_n\}.
\]

Moreover, since the Laurent series expansion of \((I + C(z))|_{\partial V}\) in a neighborhood of \(z_j\) is given by

\[
(1.12) \quad (I + C(z))|_{\partial V} = I|_{\partial V} + \sum_{k=0}^{+\infty} (z - z_j)^k C^{(j)}_k |_{\partial V},
\]

it follows that \((I + C^{(j)}_0)|_{\partial V}\) has a closed range and a trivial kernel. Using Propositions 1.4 and 1.5, we have

\[
\text{ind}(I + C^{(j)}_0) = \text{ind}(I + \sum_{k=0}^{+\infty} (z - z_j)^k C^{(j)}_k) = \text{ind}(I + C(z)) = 0.
\]

Thus, \((I + C^{(j)}_0)\) is Fredholm. By Lemma 1.11, we deduce that \(I + C(z)\) is normal with respect to \(\partial V\).

Now we claim that \(\mathcal{M}(I + C(z); \partial V) = 0\). To see this, we note that \(I + tC(z)\) is normal with respect to \(\partial V\) for \(0 \leq t \leq 1\). Let

\[
f(t) := \mathcal{M}(I + tC(z); \partial V).
\]

Then \(f(t)\) attains integers as its values. On the other hand, since

\[
(1.13) \quad f(t) = \frac{1}{2\sqrt{-1}} \text{tr} \int_{\partial V} t(I + tC(z))^{-1} \frac{d}{dz} C(z) dz
\]

and \((I + tC(z))^{-1}\) is continuous in \([0, 1]\) in operator norm uniformly in \(z \in \partial V\), \(f(t)\) is continuous in \([0, 1]\). Thus, \(f(1) = f(0) = 0\).

Finally, with the help of Corollary 1.13, we can conclude that the theorem holds. \(\square\)

**1.3.3. Generalization of Steinberg’s theorem.** Steinberg’s theorem asserts that if \(K(z)\) is a compact operator on a Banach space, which is analytic in \(z\) for \(z\) in a region \(V\) in the complex plane, then \(I + K(z)\) is meromorphic in \(V\). See \cite{238}. A generalization of this theorem to finitely meromorphic operators was first given by Gohberg and Sigal in \cite{114}. The following important result holds.

**Theorem 1.16 (Generalized Steinberg’s theorem).** Suppose that \(A(z)\) is an operator-valued function which is finitely meromorphic and of Fredholm type in the domain \(V\). If the operator \(A(z)\) is invertible at one point of \(V\), then \(A(z)\) has a bounded inverse for all \(z \in V\), except possibly for certain isolated points.
1.4. Concluding Remarks

In this chapter, we have reviewed the main results in the theory of Gohberg and Sigal on meromorphic operator-valued functions. These results concern the generalization of the argument principle and the Rouché theorem to meromorphic operator-valued functions. Some of these results have been extended to very general operator-valued functions in [46, 170] and with other types of spectrum than isolated eigenvalues in [174].

Throughout this book, the theory of Gohberg and Sigal will be applied to perturbation theory of eigenvalues. Other interesting applications include the investigation of scattering resonances and scattering poles [118, 57] and the study of the regularity of the solutions of elliptic boundary value problems near conical points [154].