Preface

This book is a revised and expanded version of the authors' manuscript "Analysis and Dynamics on the Berkovich Projective Line" ([**91**], July 2004). Its purpose is to develop the foundations of potential theory and rational dynamics on the Berkovich projective line.

The theory developed here has applications in arithmetic geometry, arithmetic intersection theory, and arithmetic dynamics. In an effort to create a reference which is as useful as possible, we work over an arbitrary complete and algebraically closed non-Archimedean field. We also state our global applications over an arbitrary product formula field whenever possible. Recent work has shown that such generality is essential, even when addressing classical problems over $\mathbb{C}$. As examples, we note the first author's proof of a Northcott-type finiteness theorem for the dynamical height attached to a nonisotrivial rational function of degree at least 2 over a function field [**5**] and his joint work with Laura DeMarco [**6**] on finiteness results for preperiodic points of complex dynamical systems.

We first give a detailed description of the topological structure of the Berkovich projective line. We then introduce the *Hsia kernel*, the fundamental kernel for potential theory (closely related to the Gromov kernel of [**47**]). Next we define a Laplacian operator on $\mathbb{P}^1_{\mathrm{Berk}}$ and construct theories of capacities, harmonic functions, and subharmonic functions, all strikingly similar to their classical counterparts over $\mathbb{C}$. We develop a theory of multiplicities for rational maps and give applications to non-Archimedean dynamics, including the construction of a canonical invariant probability measure on $\mathbb{P}^1_{\mathrm{Berk}}$ analogous to the well-known measure on $\mathbb{P}^1(\mathbb{C})$ constructed by Lyubich and by Freire, Lopes, and Mañé. Finally, we investigate Berkovich space analogues of the classical Fatou-Julia theory for rational iteration over $\mathbb{C}$.

In §7.8, we give an updated treatment (in the special case of $\mathbb{P}^1$) of the Fekete and Fekete-Szegö theorems from [**88**], replacing the somewhat esoteric notion of "algebraic capacitability" with the simple notion of compactness. In §7.9, working over an arbitrary product formula field, we prove a generalization of Bilu's equidistribution theorem [**24**] for algebraic points which are 'small' with respect to the height function attached to a compact Berkovich adelic set. In §10.3, again working over a product formula field, we prove an adelic equidistribution theorem for algebraic points which are 'small' with respect to the dynamical height attached to a rational function of degree at least 2, extending results in [**9**], [**35**], and [**47**].

A more detailed overview of the results in this book can be found in the first author's lecture notes from the 2007 Arizona Winter School [**4**], and in the Introduction below.

History

This book began as a set of lecture notes from a seminar on the Berkovich projective line held at the University of Georgia during the spring of 2004. The purpose of the seminar was to develop the tools needed to prove an adelic equidistribution theorem for small points with respect to the dynamical height attached to a rational function of degree $d \geq 2$ defined over a number field (Theorem 10.24). Establishing such a theorem had been one of the main goals in our 2002 NSF proposal DMS-0300784.

In [**8**], the first author and Liang-Chung Hsia had proved an adelic equidistribution theorem for points of $\mathbb{P}^1(\overline{\mathbb{Q}})$ having small dynamical height with respect to the iteration of a polynomial map. Two basic problems remained after that work. First, there was the issue of generalizing the main results of [**8**] to *rational functions*, rather than just polynomials. It occurs frequently in complex dynamics and potential theory that one needs heavier machinery to deal with rational maps than with polynomials. Second, because the filled Julia set in $\mathbb{P}^1(\mathbb{C}_p)$ of a polynomial over $\mathbb{C}_p$ is often noncompact, the authors of [**8**] were unable to formulate their result as a true "equidistribution" theorem. Instead, they introduced a somewhat artificial notion of "pseudo-equidistribution" and showed that when the filled Julia set is compact, then pseudo-equidistribution coincides with equidistribution.

The second author, upon learning of the results in [**8**], suggested that Berkovich's theory might allow those results to be formulated more cleanly. Several years earlier, in [**36**], he had proposed that Berkovich spaces would be a natural setting for non-Archimedean potential theory.

We thus set out to generalize the results of [**8**] to a true equidistribution theorem on $\mathbb{P}^1_{\mathrm{Berk}}$, valid for arbitrary rational maps. An important step in this plan was to establish the existence of a canonical invariant measure on $\mathbb{P}^1_{\mathrm{Berk}}$ attached to a rational function of degree at least 2 defined over $\mathbb{C}_p$, having properties analogous to those of the canonical measure in complex dynamics (see [**54, 72**]). It was clear that even defining the canonical measure would require significant foundational work.

At roughly the same time, Antoine Chambert-Loir posted a paper to the arXiv preprint server proving (among other things) non-Archimedean Berkovich space analogues of Bilu's equidistribution theorem and the Szpiro-Ullmo-Zhang equidistribution theorem for abelian varieties with good reduction. In the summer of 2003, the first author met with Chambert-Loir in Paris and learned that Chambert-Loir's student Amaury Thuillier had recently defined a Laplacian operator on Berkovich curves. Not knowing exactly what Thuillier had proved, nor when his results might be publicly available, we undertook to develop a measure-valued Laplacian and a theory of subharmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$ ourselves, with a view toward applying them in a dynamical setting. The previous year, we had studied Laplacians and their spectral theory on metrized graphs, and that work made it plausible that a Laplacian operator could be constructed on $\mathbb{P}^1_{\mathrm{Berk}}$ by taking an inverse limit of graph Laplacians.

The project succeeded, and we presented our equidistribution theorem at the conference on Arithmetical Dynamical Systems held at CUNY in May 2004. To our surprise, Chambert-Loir, Thuillier, and Pascal Autissier had proved the same theorem using an approach based on Arakelov theory. At the same conference, Rob Benedetto pointed us to the work of Juan Rivera-Letelier, who had independently rediscovered the Berkovich projective line and used it to carry out a deep study of non-Archimedean dynamics. Soon after, we learned that Charles Favre and Rivera-Letelier had independently proved the equidistribution theorem as well.

The realization that three different groups of researchers had been working on similar ideas slowed our plans to develop the theory further. However, over time it became evident that each of the approaches had merit: for example, our proof brought out connections with arithmetic capacities; the proof of Chambert-Loir, Thuillier, and Autissier was later generalized to higher dimensions; and Favre and Rivera-Letelier's proof yielded explicit quantitative error bounds. Ultimately, we, at least, have benefitted greatly from the others' perspectives.

Thus, while this book began as a research monograph, we now view it mainly as an expository work whose goal is to give a systematic presentation of foundational results in potential theory and dynamics on $\mathbb{P}^1_{\mathrm{Berk}}$. Although the approach to potential theory given here is our own, it has overlaps with the theory developed by Thuillier for curves of arbitrary genus. Many of the results in the final two chapters on the dynamics of rational functions were originally discovered by Rivera-Letelier, though some of our proofs are new.

Related works

Amaury Thuillier, in his doctoral thesis [**94**], established the foundations of potential theory for Berkovich curves of arbitrary genus. Thuillier constructs a Laplacian operator and theories of harmonic and subharmonic functions and gives applications of his work to Arakelov intersection theory. Thuillier's work has great generality and scope, but it is written in a sophisticated language and assumes a considerable amount of machinery. Because this book is written in a more elementary language and deals only with $\mathbb{P}^1$, it may be a more accessible introduction to the subject for some readers.

Juan Rivera-Letelier, in his doctoral thesis [**81**] and subsequent papers [**82, 83, 84, 80**], has carried out a profound study of the dynamics of rational maps on the Berkovich projective line (though his papers are written in a rather different terminology). Section 10.9 contains an exposition of Rivera-Letelier's work.

Using Rivera-Letelier's ideas, we have simplified and generalized our discussion of multiplicities in Chapter 9 and have greatly extended our original results on the dynamics of rational maps in Chapter 10. It should be noted that Rivera-Letelier's proofs are written with $\mathbb{C}_p$ as the ground field. One of the goals of this book is to establish a reference for parts of his theory which hold over an arbitrary complete and algebraically closed non-Archimedean field.

Charles Favre and Mattias Jonsson [**45**] have developed a Laplacian operator, and parts of potential theory, in the general context of $\mathbb{R}$-trees. Their definition of the Laplacian, while ultimately yielding the same operator on $\mathbb{P}^1_{\mathrm{Berk}}$, has a rather different flavor from ours. As noted above, Chambert-Loir [**35**] and Favre and Rivera-Letelier [**46, 47**] have given independent proofs of the adelic dynamical equidistribution theorem, as well as constructions of the canonical measure on $\mathbb{P}^1_{\mathrm{Berk}}$ attached to a rational function. Recently Favre and Rivera-Letelier [**48**] have investigated ergodic theory for rational maps on $\mathbb{P}^1_{\mathrm{Berk}}$. In Section 10.4, we prove a special case of their theorem on the convergence of pullback measures to the canonical measure, which we use as the basis for our development of Fatou-Julia theory.

Acknowledgments

We would like to thank all the people who assisted us in the course of this project, in particular our wives for their patience and understanding.

We thank Robert Varley and Mattias Jonsson for useful suggestions and Sheldon Axler, Antoine Chambert-Loir, Robert Coleman, Xander Faber, William Noorduin, Daeshik Park, Clay Petsche, Juan Rivera-Letelier, Joe Silverman, and Steve Winburn for proofreading parts of the manuscript. Aaron Abrams and Brian Conrad gave useful suggestions on Appendices B and C, respectively. Charles Favre suggested several improvements in Chapter 8. The idea for the Hsia kernel as the fundamental kernel for potential theory on the Berkovich line was inspired by a manuscript of Liang-Chung Hsia [**62**]. We thank Rob Benedetto for directing us to the work of Juan Rivera-Letelier. We also thank Rivera-Letelier for many stimulating conversations about dynamics of rational maps and the Berkovich projective line. Finally, we thank the anonymous referees who read through this work and made a number of valuable suggestions and everyone at the AMS who assisted with the production of this book, including Marcia Almeida, Barbara Beeton, and Ina Mette. We especially thank Arlene O'Sean for her careful and thorough editing.

We are grateful to the National Science Foundation for its support of this project, primarily in the form of the research grant DMS-0300784, but also through research grants DMS-0600027 and DMS-0601037, under which the bulk of the writing was carried out. Any opinions, findings, conclusions, or recommendations expressed in this work are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Differences from the preliminary version

There are several differences between the present manuscript and the preliminary version [**91**] posted to the arXiv preprint server in July 2004. For one thing, we have corrected a number of errors in the earlier version.

In addition, we have revised all of the statements and proofs so that they hold over an arbitrary complete, algebraically closed field K endowed with a nontrivial non-Archimedean absolute value, rather than just over the field $\mathbb{C}_p$. The main difference is that the Berkovich projective line over $\mathbb{C}_p$ is *metrizable* and has *countable* branching at every point, whereas in general the Berkovich projective line over K is nonmetrizable and has uncountable branching. Replacing $\mathbb{C}_p$ by K throughout required a significant reworking of many of our original proofs, since [**91**] relies in several places on arguments valid for metric spaces but not for an arbitrary compact Hausdorff space. Consequently, the present book makes more demands on the reader in terms of topological prerequisites; for example we now make use of nets rather than sequences in several places.

In some sense, this works against the concrete and "elementary" exposition that we have striven for. However, the changes seem desirable for at least two reasons. First, some proofs become more natural once the crutch of metrizability is removed. Second, and perhaps more importantly, the theory for more general fields is needed for many applications. The first author's paper [**5**] is one example of this: it contains a Northcott-type theorem for dynamical canonical heights over a general field k endowed with a product formula; the theorem is proved by working locally at each place v on the Berkovich projective line over $\mathbb{C}_v$ (the smallest complete and algebraically closed field containing k and possessing an absolute value extending the given one $|\ |_v$ on k). As another example, Kontsevich and Soibelman [**68**] have recently used Berkovich's theory over fields such as the completion of an algebraic closure of $\mathbb{C}((T))$ to study homological mirror symmetry. We mention also the work of Favre and Jonsson [**45**] on the valuative tree, as well as the related work of Jan Kiwi [**66**], both of which have applications to complex dynamics.

Here is a summary of the main differences between this work and [**91**]:

- We have added a detailed Introduction summarizing the work.
- We have added several appendices in order to make the presentation more self-contained.
- We have added a symbol table and an index and updated the bibliography.
- We give a different construction of $\mathbb{P}^1_{\mathrm{Berk}}$ (analogous to the "Proj" construction in algebraic geometry) which makes it easier to understand the action of a rational function.
- We have changed some of our notation and terminology to be compatible with that of the authors mentioned above.

- We have added sections on the Dirichlet pairing and Favre–Rivera-Letelier smoothing.
- We compare our Laplacian with those of Favre, Jonsson, and Rivera-Letelier, and of Thuillier.
- We have included a discussion of $\mathbb{R}$-trees, and in particular of $\mathbb{P}^1_{\mathrm{Berk}}$ as a "profinite $\mathbb{R}$-tree".
- We have expanded our discussion of the Poisson formula on $\mathbb{P}^1_{\mathrm{Berk}}$.
- We have added a section on Thuillier's short exact sequence describing subharmonic functions in terms of harmonic functions and positive σ-finite measures.
- We have added a section on Hartogs's lemma, a key ingredient in the work of Favre and Rivera-Letelier.
- We state and prove Berkovich space versions of Bilu's equidistribution theorem, the dynamical equidistribution theorem for small points, and the arithmetic Fekete-Szegö theorem for $\mathbb{P}^1$.
- We have simplified and expanded the discussion in Chapter 9 on analytic multiplicities.
- We have greatly expanded the material on dynamics of rational maps, incorporating the work of Rivera-Letelier and the joint work of Favre and Rivera-Letelier, and including a section on examples.