## Contents

Preface ix

Chapter 1. Background material 1
1.1. Functional analysis 1
1.2. Measures on topological spaces 6
1.3. Conditional measures 19
1.4. Gaussian measures 23
1.5. Stochastic integrals 29
1.6. Comments and exercises 35

Chapter 2. Sobolev spaces on $\mathbb{R}^n$ 39
2.1. The Sobolev classes $W^{p,k}$ 39
2.2. Embedding theorems for Sobolev classes 45
2.3. The classes $BV$ 51
2.4. Approximate differentiability and Jacobians 52
2.5. Restrictions and extensions 56
2.6. Weighted Sobolev classes 58
2.7. Fractional Sobolev classes 65
2.8. Comments and exercises 66

Chapter 3. Differentiable measures on linear spaces 69
3.1. Directional differentiability 69
3.2. Properties of continuous measures 73
3.3. Properties of differentiable measures 76
3.4. Differentiable measures on $\mathbb{R}^n$ 82
3.5. Characterization by conditional measures 88
3.6. Skorohod differentiability 91
3.7. Higher order differentiability 100
3.8. Convergence of differentiable measures 101
3.9. Comments and exercises 103

Chapter 4. Some classes of differentiable measures 105
4.1. Product measures 105
4.2. Gaussian and stable measures 107
4.3. Convex measures 112
4.4. Distributions of random processes 122
4.5. Gibbs measures and mixtures of measures 127
4.6. Comments and exercises 131

Chapter 5. Subspaces of differentiability of measures 133
5.1. Geometry of subspaces of differentiability 133
5.2. Examples 136
5.3. Disposition of subspaces of differentiability 141
5.4. Differentiability along subspaces 149
5.5. Comments and exercises 155

Chapter 6. Integration by parts and logarithmic derivatives 157
6.1. Integration by parts formulae 157
6.2. Integrability of logarithmic derivatives 161
6.3. Differentiability of logarithmic derivatives 168
6.4. Quasi-invariance and differentiability 170
6.5. Convex functions 173
6.6. Derivatives along vector fields 180
6.7. Local logarithmic derivatives 183
6.8. Comments and exercises 187

Chapter 7. Logarithmic gradients 189
7.1. Rigged Hilbert spaces 189
7.2. Definition of logarithmic gradient 190
7.3. Connections with vector measures 194
7.4. Existence of logarithmic gradients 198
7.5. Measures with given logarithmic gradients 204
7.6. Uniqueness problems 209
7.7. Symmetries of measures and logarithmic gradients 217
7.8. Mappings and equations connected with logarithmic gradients 222
7.9. Comments and exercises 223

Chapter 8. Sobolev classes on infinite dimensional spaces 227
8.1. The classes $W^{p,r}$ 227
8.2. The classes $D^{p,r}$ 230
8.3. Generalized derivatives and the classes $G^{p,r}$ 233
8.4. The semigroup approach 235
8.5. The Gaussian case 239
8.6. The interpolation approach 242
8.7. Connections between different definitions 246
8.8. The logarithmic Sobolev inequality 250
8.9. Compactness in Sobolev classes 253
8.10. Divergence 254
8.11. An approach via stochastic integrals 257
8.12. Some identities of the Malliavin calculus 263
8.13. Sobolev capacities 265
8.14. Comments and exercises 274
Contents

Chapter 9. The Malliavin calculus 279
  9.1. General scheme 279
  9.2. Absolute continuity of images of measures 282
  9.3. Smoothness of induced measures 288
  9.4. Infinite dimensional oscillatory integrals 297
  9.5. Surface measures 299
  9.6. Convergence of nonlinear images of measures 307
  9.7. Supports of induced measures 319
  9.8. Comments and exercises 323

Chapter 10. Infinite dimensional transformations 329
  10.1. Linear transformations of Gaussian measures 329
  10.2. Nonlinear transformations of Gaussian measures 334
  10.3. Transformations of smooth measures 338
  10.4. Absolutely continuous flows 340
  10.5. Negligible sets 342
  10.6. Infinite dimensional Rademacher’s theorem 350
  10.7. Triangular and optimal transformations 358
  10.8. Comments and exercises 365

Chapter 11. Measures on manifolds 369
  11.1. Measurable manifolds and Malliavin’s method 370
  11.2. Differentiable families of measures 379
  11.3. Current and loop groups 390
  11.4. Poisson spaces 393
  11.5. Diffeomorphism groups 394
  11.6. Comments and exercises 398

Chapter 12. Applications 401
  12.1. A probabilistic approach to hypoellipticity 401
  12.2. Equations for measures 407
  12.3. Logarithmic gradients and symmetric diffusions 414
  12.4. Dirichlet forms and differentiable measures 416
  12.5. The uniqueness problem for invariant measures 420
  12.6. Existence of Gibbs measures 422
  12.7. Comments and exercises 424

References 427

Subject Index 483