Preface

This book presents a fairly comprehensive account of the modern theory of quasiconformal mappings in Euclidean $n$-space for $n \geq 2$, starting from the elementary theory of conformal mappings and building towards the more general aspects by carefully developing the necessary analytic and geometric tools. This book is primarily aimed at graduate students and researchers who seek to understand quasiconformal mappings, particularly in three or more dimensions, perhaps after having seen applications of the two-dimensional theory in Teichmüller spaces of Riemann surfaces, or in conformal dynamical systems and elsewhere. However, as we carefully develop most of the necessary analytic theory only a basic background course in multi-dimensional real analysis is assumed.

The theory of quasiconformal mappings seeks to generalise the remarkable geometric and analytic theory of conformal mappings in the plane to higher dimensions. This is since Liouville’s rigidity theorem implies an extreme paucity—a finite dimensional family—of conformal mappings defined on domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Of course in two dimensions the conformal mappings of a domain form an infinite-dimensional family and one has the Riemann mapping theorem. The reasons for seeking this generalisation are manifold with wide application. For instance in the theory of partial differential equations, quasiconformal mappings preserve the ellipticity of second order equations of divergence type—those with the widest application in physics—so the solution to mapping problems enables the transfer of equations from one domain to another, potentially nicer, domain where a solution might be found.

In higher dimensions few manifolds admit a conformal structure, yet D. Sullivan has shown that every topological manifold admits a quasiconformal structure, that is, a covering with quasiconformal local coordinate charts. This presents the opportunity to compute analytic invariants on a topological manifold or to compute topological invariants analytically—for instance in the work of A. Connes, D. Sullivan, and N. Teleman. Unfortunately we will only touch on these deep applications in this work. Nevertheless the reader will find—for the first time in book form—a solid foundation to explore these remarkable results and applications.

We approach the theory of quasiconformal mappings from the geometric point of view, using conformal invariants such as the moduli of curve families and capacities. These ideas are of independent interest and again of wide utility in many areas of mathematics, and so we give a fairly thorough account of them.

We begin by developing the basics of the theory—including the study of conformal mappings in space, elementary aspects of higher-dimensional hyperbolic geometry and its isometries, along with the associated matrix groups. This leads quickly to the celebrated rigidity theorem of Liouville for smooth mappings, the proof for
which follows an argument of Nevanlinna. To get Liouville’s theorem in complete
generality, more theory—in particular the theory of conformal modulus—is developed.
The geometric aspects of the theory of quasiconformal mappings rely to a
great deal on understanding and estimating these conformal invariants. Indeed the
very definition of a quasiconformal mapping here is via the distortion of moduli by
a multiplicative factor.

We then consider deeper properties of conformal modulus such as symmetrization,
continuity, the structure of sets of capacity zero and the existence and uniqueness
of extremal functions. These give us powerful tools to study quasiconformal
mappings which enable us to not only establish analytic properties, but also to
develop the compactness and normal family properties of sequences of quasiconformal
mappings.

We then turn our attention to the mapping problem in its various forms, basically
seeking a higher-dimensional version of the Riemann mapping theorem for the
class of quasiconformal mappings. We present the classical geometric obstructions
to existence and then turn to positive results. We give a proof for the Schoenflies
theorem in the quasiconformal category and subsequently give a fairly complete
proof of Väisälä’s mapping theorem for cylindrical domains, perhaps the best re
sult to date answering this question.

We then present the sophisticated and important work of Tukia-Väisälä developed
using Sullivan’s machinery. In particular we give a proof for their solution of
the lifting problem. Many of the last chapters of this book—part of a central theme
in the area to develop quasiconformal versions of classical theorems in geometric
topology—have never previously appeared in book form. Indeed many aspects of
the approach to the theory given here are novel among recent monographs on the
subject, as these primarily focus on the analytic approach through the associated
nonlinear partial differential equations and differential inequalities.

We close with a presentation of the Mostow rigidity theory, one of the most
compelling and important applications of the higher-dimensional theory of quasi
conformal mappings. We take a fairly roundabout approach here so as to be able
to clearly exhibit the remarkable interaction between quasiconformal theory, hyperbolic geometry, and modern aspects of geometric group theory. In particular
we give a fairly comprehensive discussion of quasi-isometries and isomorphisms of
hyperbolic groups.

During the long gestation of this book the first-named author Fred Gehring
passed away. He was of course a major figure in the area, and much of the important
work presented in this book is due to him and his coauthors. He is sadly missed.

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Gaven Martin and Bruce Palka