Reminders

We briefly recall our conventions before giving an overview of the contents of this volume.

The notation of categories. First recall that we use calligraphic letters, like $\mathcal{C}$, $\mathcal{D}$, ..., $\mathcal{M}$, $\mathcal{N}$, ... as a generic notation for the categories in which we define our objects. The most fundamental examples of categories which we consider in this book include the category of modules over a fixed ground ring $k$, which we denote by $\mathcal{C} = \mathsf{Mod}$, the category of topological spaces $\mathcal{C} = \mathsf{Top}$, and the category of simplicial sets $\mathcal{C} = \mathsf{sSet}$ (where the plain notation $\mathsf{Set}$ refers to the category of sets).

We use the notation $\mathsf{dgMod}$ for the category of differential graded modules over $k$ (which we also call $\mathsf{dg}$-modules for short), the notation $\mathsf{grMod}$ for the category of graded modules (which we regard as the subcategory of $\mathsf{dg}$-modules equipped with a trivial differential) and the notation $\mathsf{sMod}$ for the category of simplicial modules (which we define as the category of simplicial objects in $\mathsf{Mod}$). We review the precise definition of these categories in the course of this volume. We also recall the definition of a chain (respectively, cochain) graded variant of the category of $\mathsf{dg}$-modules $\mathsf{dg}^* \mathsf{Mod}$ (respectively, $\mathsf{dg}^* \mathsf{Mod}$) in our study of the rational homotopy of spaces. We still consider a category of cosimplicial modules $\mathsf{cMod}$ which we define, dually to the category of simplicial modules, as the category of cosimplicial objects in $\mathsf{Mod}$.

We use expressions with a calligraphic capital (like $\mathcal{P} = \mathcal{C}om$, $\mathcal{O}p$, ...) for the categories of structured objects (commutative algebras, operads, ...) which we may form in any of these base categories. We just add the base category $\mathcal{M} = \mathsf{Top}, \mathsf{sSet}, \ldots, \mathsf{Mod}, \ldots$ as a prefix to the notation $\mathcal{P} = \mathcal{M} \mathcal{P}$ of any such category $\mathcal{P} = \mathcal{C}om, \mathcal{O}p, \ldots$ when this precision is necessary. The notation $\mathcal{C}om = \mathsf{dgMod} \mathcal{C}om$, for instance, refers to the category of commutative algebras $\mathcal{P} = \mathcal{C}om$ in the base category of $\mathsf{dg}$-modules $\mathcal{M} = \mathsf{dgMod}$.

We withdraw the expression $\mathsf{Mod}$ and reduce our notation of the base category to the prefix $\mathcal{P} = \mathsf{dg}, \mathsf{gr}, \mathsf{s}, \ldots$ when we deal with a category of structured objects $\mathcal{P} = \mathcal{C}om, \mathcal{O}p, \ldots$ in any of our variants $\mathcal{M} = \mathsf{dgMod}, \mathsf{grMod}, \ldots$ of the category of ordinary modules $\mathsf{Mod}$. We accordingly use the notation $\mathsf{dgC}om$ (respectively, $\mathsf{grC}om$, $\mathsf{sC}om$, ...) for the category of commutative algebras in $\mathsf{dg}$-modules (respectively, in graded modules, in simplicial modules, ...), and we adopt similar conventions in the case of operads.

Recall that $\mathcal{C}om$ is actually our notation for the category of commutative algebras without unit. In what follows, we mostly deal with unitary commutative algebras and we adopt the notation $\mathcal{C}om_+$, with the extra postfix subscript $+$, to refer to this category. We also use the notation $\mathcal{C}om^c_+$, with the extra postfix superscript $c$, for the category of counitary cocommutative algebras in any base category.
(see §I.3.0 for the general definition of this notion). We go back to our conventions concerning operads later on in this reminder section.

**Symmetric monoidal structures.** In the first volume, we explain that we use a symmetric monoidal structure given with our base category $\mathcal{M} = \mathcal{O}p, s\mathcal{S}et, \ldots$ to define our categories of unitary commutative algebras, of operads, $\ldots$. Briefly recall that this symmetric monoidal structure is defined by a tensor product bifunctor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, together with a unit object, which we denote by $1 \in \mathcal{M}$, and with natural associativity relations which give the unit relations $X \otimes 1 \simeq X \simeq 1 \otimes X$, the associativity relations $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ and the symmetry relations $X \otimes Y \simeq Y \otimes X$ in $\mathcal{M}$.

In the case $\mathcal{M} = \mathcal{S}et, \mathcal{O}p, s\mathcal{S}et, \ldots$, we take the cartesian product $\otimes = \times$ to define the symmetric monoidal structure of our base category. In the case $\mathcal{M} = \mathcal{M}od$, we take the usual tensor product of modules over the ground ring $\otimes = \otimes_k$. We also use this standard tensor product to define a tensor product operation on the category of dg-modules $\mathcal{M} = \mathcal{M}od$ (and of graded modules $\mathcal{M} = \mathcal{G}rad\mathcal{M}od$) but we modify the ordinary symmetry isomorphism in the dg-module setting in order to implement the permutation rules of differential graded algebra in the symmetric monoidal structure of our category (see §0.2). We explain the definition of this symmetric monoidal category of dg-modules with full details in §II.5.2. We explain the definition of a symmetric monoidal structure on the category of simplicial modules and on the category of cosimplicial modules in §II.5.2 too. We use these symmetric monoidal structures in order to formalize the definition of unitary commutative algebras in dg-modules, in simplicial modules, and in cosimplicial modules (see §II.6.1).

Recall that a tensor product of unitary commutative algebras inherits a natural unitary commutative algebra structure so that the category of unitary commutative algebras $\mathcal{C}om_+ = \mathcal{M} \mathcal{C}om_+$ in a base symmetric monoidal category $\mathcal{M}$ forms a symmetric monoidal category itself. Recall also that we have a similar assertion for the categories of counitary cocommutative coalgebras $\mathcal{C}om^+_c = \mathcal{M} \mathcal{C}om^+_c$. (see §I.3.0.4). We use these symmetric monoidal structures to formalize the definition of our notion of a Hopf operad and of a Hopf cooperad. To be explicit, we define a Hopf operad as an operad in the symmetric monoidal category of counitary cocommutative coalgebras and we define a Hopf cooperad as a cooperad in the symmetric monoidal category of unitary commutative algebras. We go back to this subject in §II.9.

We often assume that the tensor product of our base category $\mathcal{M}$ distributes over colimits in the sense that we have the relation $(\operatorname{colim}_\alpha X_\alpha) \otimes Y \simeq \operatorname{colim}_\alpha (X_\alpha \otimes Y)$ for any diagram of objects $X_\alpha, \alpha \in J$, in the category $\mathcal{M}$ and for any fixed object $Y \in \mathcal{M}$. We symmetrically have $X \otimes (\operatorname{colim}_\beta Y_\beta) \simeq \operatorname{colim}_\beta (X \otimes Y_\beta)$ when we fix $X \in \mathcal{M}$, and we take the colimit of a diagram of objects $Y_\beta, \beta \in J$, on the right-hand side (see §0.9). The cartesian product satisfies these relations in the category of sets $\mathcal{M} = \mathcal{S}et$, topological spaces $\mathcal{M} = \mathcal{O}p$, and simplicial sets $\mathcal{M} = s\mathcal{S}et$, and so do the usual tensor product of the category of modules over our ground ring $\mathcal{M} = \mathcal{M}od$, the tensor product of the category of dg-modules $\mathcal{M} = \mathcal{M}od$, $\ldots$.

The tensor product of counitary cocommutative coalgebras distributes over colimits as well because the colimits of counitary cocommutative coalgebras are created in the base category (see §I.3.0.4). We see on the other hand that the tensor product of unitary commutative algebras does not distribute over coproducts (and
hence, over colimits in general) since the tensor product also realizes coproducts in this category (see §I.3.0.3). In this case, we will rather assume that the tensor product distributes over finite limits. We go back to this idea in §II.9.1 when we explain the definition of our category of Hopf cooperads.

To define generalizations of the Postnikov decomposition for operads, we also consider the direct sum operation $\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ as the tensor product operation of an additive symmetric monoidal structure on our module categories $\mathcal{M} = \text{dgMod}, \text{sMod}, \ldots$ (see §III.1.2). This tensor product $\otimes = \oplus$ obviously does not distribute over colimits either.

**Morphisms, homomorphisms, hom-objects and duals.** We adopt the notation $\text{Mor}_\mathcal{C}(X, Y)$ for the set of morphisms associated to any pair of objects $X, Y \in \mathcal{C}$ in a category $\mathcal{C}$.

Many categories which we consider in this book are also endowed with the structure of an enriched category over a base category. To be explicit, we often assume that our category $\mathcal{C}$ is equipped with a hom-object bifunctor, which we usually denote by $\text{Hom}_\mathcal{C}(\_\_ : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{M}$, together with a unit operation $\eta : 1 \to \text{Hom}_\mathcal{C}(X, X)$, defined for any $X \in \mathcal{C}$, and a composition operation $\circ : \text{Hom}_\mathcal{C}(Z, Y) \otimes \text{Hom}_\mathcal{C}(X, Z) \to \text{Hom}_\mathcal{C}(X, Y)$ which mimics the unit and the composition operations of the ordinary morphisms. We use the name homomorphism to distinguish the elements of these hom-objects $f \in \text{Hom}_\mathcal{C}(X, Y)$ from the actual morphisms of our category, which are the elements of the morphism sets $\text{Mor}_\mathcal{C}(X, Y)$ (see §0.13 for more details explanations on this convention).

In our study, we also consider simplicial mapping spaces $\text{Map}_\mathcal{C}(X, Y)$ whose homotopy determines the morphism sets of the homotopy category of model categories (see §§II.2-3). In the case of a simplicial model category (see §II.2), we assume that these mapping spaces define the hom-objects of an enriched category structure with values in the category of simplicial sets, but in general, we just define such mapping spaces by homotopy theory constructions (see §II.3) and we do not have strict composition operations (and hence, a strict enriched category structure) on these objects.

In the case $\mathcal{M} = \text{Top}, \text{sSet}, \ldots, \text{Mod}, \ldots$, we generally use that our category $\mathcal{C}$ is equipped with a closed symmetric monoidal structure to get that $\mathcal{M}$ is enriched over itself. We explicitly have an internal hom-object bifunctor $\text{Hom}_\mathcal{M}(\_\_ : \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{M}$ which we characterize by the adjunction relation $\text{Mor}_\mathcal{M}(X \otimes Y, Z) = \text{Mor}_\mathcal{M}(X, \text{Hom}_\mathcal{M}(Y, Z))$ in our category (see §0.14). (We just make the choice of a convenient category of topological spaces in order to ensure that such an adjunction relation holds in the case $\mathcal{M} = \text{Top}$.) In the case $\mathcal{M} = \text{Mod}$, we actually have $\text{Hom}_\text{Mod}(\_\_, -) = \text{Mor}_\text{Mod}(\_\_, -)$ since the morphism set $\text{Mor}_\text{Mod}(X, Y)$ associated to any pair of modules $X, Y \in \text{Mod}$ inherits a natural module structure and satisfies this adjunction relation $\text{Mor}_\text{Mod}(X \otimes Y, Z) = \text{Mor}_\text{Mod}(X, \text{Hom}_\text{Mod}(Y, Z))$.

We can also use the mapping $D : M \mapsto \text{Hom}_\mathcal{M}(M, 1)$, where we consider a hom-object with values in the unit object $1 \in \mathcal{M}$ of our symmetric monoidal structure to define a natural duality functor $D : \mathcal{M}^{\text{op}} \to \mathcal{M}$ on our base category $\mathcal{M}$. We just get the standard duality functor $D(M) = \text{Hom}_\text{Mod}(M, k)$ when we work in the category of modules $\mathcal{M} = \text{Mod}$ over our ground ring $k$. We examine the definition of duality functors on the category of dg-modules $\mathcal{M} = \text{dgMod}$, of simplicial modules $\mathcal{M} = \text{sMod}$ and of cosimplicial modules $\mathcal{M} = c\text{Mod}$ with full details in §II.5. We will see that we can use this hom-object construction $D(M) = \text{Hom}_{\text{dgMod}}(M, k)$, where we
regard the ground ring $k$ as a unit object for the tensor product of dg-modules, to get an internal duality functor on the category of dg-modules $D: Mod^{op} \rightarrow Mod$, while we rely on the duality of plain modules $D: Mod^{op} \rightarrow Mod$ to define duality functors $D: s Mod^{op} \rightarrow c Mod$ and $D: c Mod^{op} \rightarrow s Mod$ which exchange the category of simplicial modules $M = s Mod$ and the category of cosimplicial modules $M = c Mod$.

In what follows, we generally use the notation $D: M \mapsto D(M)$ to refer to these duality functors on our base categories. We also use the notation $M^\vee = D(M)$ for the image of individual objects $M \in M$ under such duality functors and when we deal with objects equipped with extra structures (like product or coproduct operations).

The notion of an operad. We mostly use the definition of operads in terms of partial composition operations in this volume. Recall that we tacitly assume that our operads are not defined in arity zero. We say by all natural numbers $r \in \mathbb{N}$.

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category of all operads generated by the objects \( P \in \mathcal{O} \) which have the initial object of our base category as component of arity zero. In the generic case, we use the notation \( \emptyset \in \mathcal{M} \) for this initial object. Hence, the expression \( \mathcal{P}(0) = \emptyset \) also reflects this category embedding \( \mathcal{O}_\mathcal{P} \to \mathcal{O} \) in the case where the tensor product of our ambient symmetric monoidal distributes over colimits.

We also say that a (non-unitary) operad \( \mathcal{P} \) is connected when we have the relation \( \mathcal{P}(1) = 1 \) (in addition to \( \mathcal{P}(0) = \emptyset \)) and the unit morphism of our operad \( \eta : 1 \to \mathcal{P}(1) \) reduces to the identity morphism of the unit object of our ambient symmetric monoidal category \( 1 \in \mathcal{M} \). We use the notation \( \mathcal{O}_\mathcal{P}_\emptyset 1 \) for the full subcategory of the category of non-unitary operads generated by the connected (non-unitary) operads.

In the context of non-unitary operads (respectively, of connected operads), we also consider the category \( \mathcal{S}_{\mathcal{O}} > 0 \) (respectively, \( \mathcal{S}_{\mathcal{O}} > 1 \)) formed by the symmetric sequences which are only defined in arity \( r > 0 \) (respectively, in arity \( r > 1 \)). We call non-unitary symmetric sequences (respectively, connected symmetric sequences) the objects of this category. We can still identify this category of non-unitary symmetric sequences (respectively, of connected symmetric sequences) with the full subcategory of the category of all symmetric sequences generated by the objects such that \( \mathcal{M}(0) = \emptyset \) (respectively, \( \mathcal{M}(0) = \mathcal{M}(1) = \emptyset \)) when necessary.

The first example of an operad which we consider in this book is thenon-unitary operad of commutative algebras (the commutative operad). This operad, which we denote by \( \text{Com} \), can be defined in any symmetric monoidal \( \mathcal{M} \). We explicitly have \( \text{Com}(r) = 1 \), for every \( r > 0 \), where we consider the unit object of our category \( 1 \in \mathcal{M} \) (see §I.2.1.11).

Unitary operads. Besides the category of non-unitary operads, of which we recall the definition in the previous paragraph, we also consider a category of operads \( \mathcal{P} \) which satisfy \( \mathcal{P}(1) = 1 \) in arity zero. We say that \( \mathcal{P} \) is a unitary operad when this condition holds (the terminology ‘unital operad’ is used for such objects in [118], but the name ‘unital’ is also used to refer to the unit of our operads in other references, and we therefore prefer to introduce another name to avoid confusion).

We use the notation \( \mathcal{O}_\mathcal{P}_* \) for the subcategory which has the unitary operads as objects and the operad morphisms which reduce to the identity of the unit object in arity one as morphisms. We actually use the notation \( * \) in any concrete symmetric monoidal in order to distinguish the arity zero element \( * \in \mathcal{P}(0) \), which we can associate to the unit object \( \mathcal{P}(0) = 1 \) in the arity zero component of a unitary operad, from the operadic unit \( 1 \in \mathcal{P}(1) \). In what follows, we also consider a category of connected unitary operads \( \mathcal{O}_\mathcal{P}_* \subset \mathcal{O}_\mathcal{P} \) formed by the objects \( \mathcal{P} \in \mathcal{O}_\mathcal{P}_* \) which satisfy the connectedness condition \( \mathcal{P}(1) = 1 \) in addition to the relation \( \mathcal{P}(0) = 1 \).

The notion of an (augmented) non-unitary \( \Lambda \)-operad. In subsequent constructions, we crucially use that the category of unitary operads \( \mathcal{O}_\mathcal{P}_* \) is equivalent to a category formed by non-unitary operads \( \mathcal{P} \in \mathcal{O}_\mathcal{P}_\emptyset \) equipped with extra operations which model the operadic composition with an extra unit object \( \mathcal{P}(0) = 1 \) in a unitary extension \( \mathcal{P}_+ \in \mathcal{O}_\mathcal{P}_* \) of our operad \( \mathcal{P} \). (To be precise, we only use this correspondence in our study of unitary operads. We therefore suggest the reader who is not interesting by this setting to skip the reminders of this paragraph in a first reading.)
We consider the category, denoted by $\Lambda$, which has the finite ordinals $r = \{ 1 < \cdots < r \}$ as objects and all injective maps between such ordinals as morphisms. Recall that any map $f \in \text{Mor}_\Lambda(m,n)$ in this category $\Lambda$ has a unique decomposition $f = us$, where $u : \{ 1 < \cdots < m \} \to \{ 1 < \cdots < n \}$ is a non-decreasing injection and $s : \{ 1 < \cdots < m \} \to \{ 1 < \cdots < m \}$ is a bijection of the set $m = \{ 1 < \cdots < m \}$ which is also equivalent to a permutation on $m$ letters $s = (s(1), \ldots, s(m)) \in \Sigma_m$. To express this property, we also say that the category $\Lambda$ has a decomposition $\Lambda = \Lambda^+ \Sigma$, where $\Lambda^+$ denotes the subcategory of $\Lambda$ with the same objects as $\Lambda$ but the non-decreasing injections as morphisms, and $\Sigma$ refers to the disjoint union of the symmetric groups $\Sigma_n$ (regarded as categories with a single object) in the category of categories. For our purpose, we also consider the full subcategory $\Lambda_{>0}$ (respectively, $\Lambda_{>1}$) of the category $\Lambda$ generated by the ordinals $r = \{ 1 < \cdots < r \}$ of cardinal $r > 0$ (respectively, $r > 1$). We still have the relation $\Lambda_{>0} = \Lambda^+ \cap \Lambda_{>0}$ and $\Sigma_{>0} = \Lambda^+ \cap \Lambda_{>0}$ (and we similarly have $\Lambda_{>1} = \Lambda^+ \cap \Sigma_{>1}$ in the case of the category $\Lambda_{>1} \subset \Lambda$).

We precisely call augmented non-unitary $\Lambda$-operad the structure defined by a collection $P = \{ P(r), r > 0 \}$ equipped with the structure of a contravariant diagram over the category $\Lambda_{>0}$ together with an augmentation $\epsilon : P \to \text{Cst}$ over the constant diagram $\text{Cst}(r) = 1$, a unit morphism $\eta : \emptyset \to P(1)$ and composition products $c_i : P(m) \otimes P(n) \to P(m+n-1)$ such that a natural extension of the equivariance, unit and associativity relations of operads hold. We usually use the notation $u^* : P(n) \to P(m)$ for the action of the morphisms $u \in \text{Mor}_\Lambda(m,n)$ of the category $\Lambda_{>0} \subset \Lambda$ on our object $P$ and we refer to these morphisms $u^* : P(n) \to P(m)$ as the restriction operators (or the restriction operations) associated to our operad.

We observed in §I.2.2 that the non-unitary operad of commutative algebras $\text{Com}$ inherits the structure of an augmented non-unitary $\Lambda$-operad and the constant diagram $\text{Cst}(r) = 1$ in our definition actually represents the contravariant $\Lambda$-diagram underlying this operad $\text{Com}$. We therefore use the notation of this operad $\text{Com}$ rather than the notation of the constant diagram $\text{Cst}$ in our subsequent applications of the definition of an augmented non-unitary $\Lambda$-operad. We also observed in §I.2.2 that the augmentation morphism of an augmented non-unitary $\Lambda$-operad actually forms a morphism of augmented non-unitary $\Lambda$-operads $\epsilon : P \to \text{Com}$. The commutative operad $\text{Com}$ therefore represents the terminal object of the category of augmented $\Lambda$-operads.

In general, we use the notation $\Lambda \, \text{Op}_\varnothing / \text{Com}$ for the category of augmented non-unitary $\Lambda$-operads in a base symmetric monoidal category $\mathcal{M}$. But, on the other hand, we can forget about the augmentation when we work in a symmetric monoidal category (such as $\mathcal{M} = \text{Top}, s\text{Set}, \ldots$) where the unit object $1$ is identified with the terminal object $\ast \in \mathcal{M}$. In this case, we shorten our terminology to ‘non-unitary $\Lambda$-operad’ for the objects of our category of augmented non-unitary $\Lambda$-operad and we use the abridged notation $\Lambda \, \text{Op}_\varnothing = \Lambda \, \text{Op}_{\varnothing} / \text{Com}$ for this category of operads. We also use this convention in the case of (augmented) non-unitary Hopf $\Lambda$-operads which we define as (augmented) non-unitary $\Lambda$-operads in counitary cocommutative coalgebras.

The equivariance relations of the composition products of an augmented $\Lambda$-operad actually divide in two classes, which involve (partially defined) operadic composition operations on the morphisms of our category $\Lambda_{>0} \subset \Lambda$ in the first
case, and operadic composition operations with the empty maps \( o \in \text{Mor}_\Lambda(\emptyset, n) \), with the empty ordinal \( \emptyset \) as domain, in the second case. In the context of a concrete symmetric monoidal category, the first class of equivariance relations read \((u \circ_{u(i)} v)^* (p \circ_{u(i)} q) = u^* (p) \circ_i v^*(q)\), for any pair of operad elements \( p \in P(m), q \in P(n) \), for any pair of injective maps \( u \in \text{Mor}_\Lambda(k, m) \), \( v \in \text{Mor}_\Lambda(l, n) \), for any composition index \( i \in \{1 \leq \cdots < k\} \), and where \( u \circ_{u(i)} v \in \text{Mor}_\Lambda(k + 1 - 1, m + n - 1) \) denotes the alluded to operadic composite of our maps \( u \) and \( v \) in the category \( \Lambda \). The second class of equivariance relations read \((u \circ_{u(i)} o)^* (p \circ_{u(i)} q) = \epsilon(q) \cdot u^* \partial_i (p)\), for any pair of operad elements \( p \in P(m), q \in P(n) \), any injective maps \( u \in \text{Mor}_\Lambda(k, m) \), and any composition index \( i \in \{1 \leq \cdots < k\} \), where we consider the empty map \( o \in \text{Mor}_\Lambda(\emptyset, n) \), the restriction operator \( \partial_i : P(m) \rightarrow P(m - 1) \) associated to the increasing map \( \partial^i \in \text{Map}_\Lambda(m - 1, m) \) which skips the value \( i \in \{1, \ldots, m\} \) in the ordinal \( m = \{1 \leq \cdots < m\} \), and the operadic composite \( u \circ_{u(i)} o \in \text{Mor}_\Lambda(k - 1, m + n - 1) \) of the map \( u \) with the empty map \( o \). We refer to §I.2.2.12 for the explicit definition of these operadic composition operations of the maps of the category \( \Lambda \). (We also review the definition of these operations with full details in §II.11.1.2 when we explain the definition of the structure of a coaugmented \( \Lambda \)-cooperad dual to our augmented \( \Lambda \)-cooperads.) We moreover have the relations \( \epsilon(p \circ_i q) = \epsilon(p) \epsilon(q) \) when we apply the augmentation \( \epsilon : P(m + n - 1) \rightarrow 1 \) to a composite operation \( p \circ_i q \in P(m + n - 1) \).

The correspondence between unitary operads and augmented non-unitary \( \Lambda \)-operads. Recall that we call \textit{unitary extension} of a non-unitary operad \( P \) any unitary operad \( P_+ \) such that \( P_+(0) = 1 \) and \( P_+(r) = P(r) \) for \( r > 0 \). The restriction operators \( u^* : P(n) \rightarrow P(m) \) in the definition of an augmented \( \Lambda \)-operad actually model substitution operations \((u^* p)(x_1, \ldots, x_m) = p(y_1, \ldots, y_n)\) such that

\[
y_j = \begin{cases} x_{u(i)}, & \text{if } j \in \{u(1), \ldots, u(m)\}, \\ * & \text{otherwise}, \end{cases}
\]

where * refers to the distinguished element of the unitary operad \( P_+ \) which we associate to the extra unit object \( P_+(0) = 1 \) in arity zero. The augmentations \( \epsilon : P(r) \rightarrow 1 \) represent the full substitution operation \( \epsilon(p) = p(*, \ldots, *) \). The equivariance relations recalled in the previous paragraph reflect the distribution of arity zero elements * which we get in the composition of operations of our operad. From this correspondence, we tautologically get that the mapping \( P \mapsto P_+ \) defines an isomorphism of categories \( \Lambda \text{Op}_{\com} / \com \simeq \text{Op}_{P_*} \) from the category of augmented non-unitary \( \Lambda \)-operads \( \Lambda \text{Op}_{\com} / \com \) to the category of unitary operads \( \text{Op}_{P_*} \).

We have an analogous isomorphism of categories \( \Lambda \text{Op}_{\com 1}/ \com \simeq \text{Op}_{P_{*+1}} \) when we consider the category of connected unitary operads \( \text{Op}_{P_{*+1}} \) and a category of augmented connected \( \Lambda \)-operads \( \Lambda \text{Op}_{\com 1}/ \com \) for which we assume the relation \( P(1) = 1 \) in addition to \( P(0) = \emptyset \). In what follows, we use the notation \( \Lambda \text{Seq}_{\geq 0} \) for the category of contravariant \( \Lambda_{>0} \)-diagrams underlying the structure of an augmented non-unitary \( \Lambda \)-operad. We also call \textit{non-unitary \( \Lambda \)-sequence} the objects of this category of diagrams. In the case where we deal with augmented connected \( \Lambda \)-operads, we also consider a category of contravariant \( \Lambda_{>1} \)-diagrams, which we denote \( \Lambda \text{Seq}_{>1} \). We call \textit{connected \( \Lambda \)-sequence} the objects of this category of diagrams. We moreover consider the category \( \Lambda \text{Seq}_{\geq 0}/ \com \) formed by the non-unitary \( \Lambda \)-sequences equipped with an augmentation over the constant diagram \( \com(r) = 1 \) underlying the commutative operad \( \com \), and the category
$\Lambda \text{Seq}_{\geq 1}/\text{Com}$ formed by the connected $\Lambda$-sequences equipped with an augmentation over the augmentation ideal of the commutative operad $\text{Com}$. Recall that this object $\text{Com}$ is the connected symmetric sequence which we obtain by forgetting about the term of arity one $\text{Com}(1) = 1$ of the commutative operad $\text{Com}$.

**Symmetric collections and operads shaped on the category of finite sets.** In what follows, we most usually assume that the components of an operad $P(r)$ are indexed by natural numbers $r \in \mathbb{N}$ (or by positive natural numbers $r > 0$ in the context of non-unitary operads). Intuitively, we assume that an operad $P$ collects operations $p = p(x_1, \ldots, x_r)$ on variables indexed by the finite ordinals $r = \{1 < \cdots < r\}$ when we use this convention.

For certain constructions however it is more convenient to allow operads whose components are indexed by arbitrary finite sets $r$. We then consider the category $\mathcal{B}ij$ formed by the finite sets as objects and the bijective maps of finite sets as morphisms. The category of symmetric sequences is equivalent to the category of (covariant) functors over this category $\mathcal{B}ij$, and an operad in the ordinary sense is equivalent to an object of this category of (covariant) functors equipped with a unit morphism $\eta : 1 \to P(1)$, where $1$ denotes the one-point set in the category $\mathcal{B}ij$, and with composition morphisms $\circ_i : P(m) \otimes P(n) \to P(m \circ_i n)$, defined for all finite sets $m = \{i_1, \ldots, i_m\}$, $n \in \{j_1, \ldots, j_n\}$, for each composition index $i_k \in m$, and with values in the component of our operad associated to the finite set such that $m \circ_i n = \{i_1, \ldots, \widehat{i_k}, \ldots, i_m\} \amalg \{j_1, \ldots, j_n\}$. We still obviously assume that these composition morphisms shaped on the category of finite sets fulfill natural equivariance, unit, and associativity relations. We refer to §I.2.5 for a detailed survey of this definition of an operad. We also consider the full subcategory $\mathcal{B}ij_{>0} \subset \mathcal{B}ij$ (respectively, $\mathcal{B}ij_{>1} \subset \mathcal{B}ij$) generated by the finite sets $r \in \text{Ob} \mathcal{B}ij$ of cardinal $r > 0$ (respectively, $r > 1$).

We have an analogous extension of our definitions in the context of augmented non-unitary $\Lambda$-operads. We then consider the category $\mathcal{I}nj$ formed by the finite sets as objects and the injective maps between finite sets as morphisms. We also consider the full subcategory of this category $\mathcal{I}nj_{>0} \subset \mathcal{I}nj$ (respectively, $\mathcal{I}nj_{>1} \subset \mathcal{I}nj$) generated by the finite sets $r \in \text{Ob} \mathcal{I}nj$ of cardinal $r > 0$ (respectively, $r > 1$). We get that an augmented non-unitary $\Lambda$-operad is equivalent to a contravariant diagram over the category $\mathcal{I}nj_{>0}$ equipped with a composition structure shaped on this category. We also go back to this correspondence in the dual context of cooperads in §II.11.1.6.

We mainly use the indexing by arbitrary finite sets in our study of free operads (see §§A-B) and of cofree cooperads (see §C).
Reading Guide and Overview of the Volume

Recall that this monograph comprises three main parts: Part I, “From Operads to Grothendieck–Teichmüller Groups” (in the first volume), which is mainly devoted to the algebraic foundations of our subject; Part II, “Homotopy Theory and its Applications to Operads”, where we develop our rational homotopy theory of operads after a comprehensive review of the applications of methods of homotopy theory; and Part III, “The Computation of Homotopy Automorphism Spaces of Operads”, where we work out our problem of giving a homotopy interpretation of the Grothendieck–Teichmüller group (the ultimate goal of this work).

These parts are widely independent from each others (as we explained in volume one). Recall also that each part of this book is divided into subparts which, by themselves, form self-contained groupings of chapters, devoted to specific topics, and organized according to an internal progression of the level of the chapters each. There is a progression in the level of the parts of the book too, but the chapters are written so that a reader with a minimal background could tackle any of these subparts straight away in order to get a reference and a self-contained overview of the literature on each of the subjects addressed in this monograph.

This volume comprises the second named parts of the book, “Part II: Homotopy Theory and its Applications to Operads”, “Part III: The Computation of Homotopy Automorphism Spaces of Operads”, and “Appendix C: Cofree Cooperads and the Bar Duality of Operads”.

The following overview is not intended for a linear reading but should serve as a guide each time the reader tackles new parts of this volume.

**Part II. Homotopy Theory and its Applications to Operads.** The second part of this book includes: an introduction to the concepts of the theory of model categories and its applications in homotopy; a detailed review of the rational homotopy theory, from the algebraic background of the subject to the definition of models for the rational homotopy of spaces; a new definition of a model for the rational homotopy of operads; and a study of the applications of this model to $E_n$-operads.

**Part II(a). General Methods of Homotopy Theory.** We give an introduction to the general applications of the theory of model categories in this part. Most of the ideas explained in this part are not original. Nevertheless we will provide a detailed account of some particular results, which are certainly well known for the experts of the domain, but for which we can hardly get a reference. The first chapter of the part (§1) is an introductory survey of the axioms of model categories and of the construction of homotopy categories in the model category framework. The second and third chapters (§§2-3) are devoted to the applications of simplicial structures in model categories (the definition of general mapping spaces, together
with the definition of generalizations of the classical geometric realization of simplicial sets and of the totalization of cosimplicial spaces). By the way, we also explain the general definition of a homotopy automorphism space in this second chapter. The fourth chapter of the part (§4) is a survey (mostly without proofs) of the definition of the notion of a cofibrantly generated model category, an abstract setting where we have an analogue of the classical cell approximations of topology. We heavily use cofibrantly generated model structures to define the model categories which we consider in our study of the rational homotopy theory.

Chapter 1. Model Categories and Homotopy Theory. In a preliminary section of this chapter (§1.0), we explain the general problem of defining the localization of a category with respect to a class of weak-equivalences. We make explicit the axioms of model categories afterwards (in §1.1). In brief, the main idea of the theory of model categories is to use two auxiliary classes of morphisms, called cofibrations and fibrations, which are endowed with lifting properties similar to the properties of the classical cofibrations and fibrations of topology, in order to handle the definition of the morphism sets of the localization of our category. We will more precisely see that the localization of a model category is given by a homotopy category, which we construct by using the extra structure of the cofibrations and fibrations of our model category and by generalizing the definition of the classical homotopy category of spaces (§1.2).

We then review the classical definitions of fundamental model structures on topological spaces and simplicial sets (§1.3). We conclude this chapter by a brief account of the definition of model category structures on operads and on the categories of algebras associated to operads (§1.4).

The purpose of this chapter is only to give an introductory survey of the subject of model categories and to recall the most fundamental definitions of the theory which we use all through this volume. We therefore omit (or abridged) most proofs and we refer to the literature of the domain for details in general.

Chapter 2. Mapping Spaces and Simplicial Model Categories. The main purpose of this chapter is to explain the definition of the concept of a simplicial model category, where we have simplicial mapping spaces which give a generalization of the classical mapping spaces of the category of topological spaces.

We devote a short preliminary section of the chapter to the determination of functors on simplicial sets from cosimplicial objects (§2.0). We use this correspondence to relate the mapping spaces (the hom-objects) of the structure of an enriched category over simplicial sets to tensor product operations (and function objects) over the category of simplicial sets. We then explain the axioms of a simplicial model category, which ensure that the mapping spaces of such an enriched category structure satisfy appropriate homotopy properties. We check that, in the context of simplicial model categories, the homotopy of the simplicial mapping spaces determine the morphisms sets of the homotopy category associated to our model category (§2.1). This section is only a survey of classical ideas which we recall for the sake of reference.

We explain the definition of homotopy automorphism spaces in simplicial model categories afterwards (§2.2). We notably check that homotopy automorphism spaces define homotopy invariant simplicial monoids associated to the objects of our model category. This statement is known to experts, but we can hardly find a detailed proof of this observation in the literature.
We conclude this chapter by an overview of the definition of simplicial model structures for operads and for the categories of algebras associated to operads (§2.3).

Chapter 3. Simplicial Structures and Mapping Spaces in General Model Categories. This chapter is a continuation of the study initiated in the previous chapter. Our first purpose is to explain the construction of simplicial mapping spaces in general model categories. The general simplicial mapping spaces do not inherit strictly defined composition operations, in contrast to the hom-objects of a category enriched over simplicial sets, but we will explain that these objects can still be used to compute the morphism sets of the homotopy category in general model categories. To achieve this goal, we first review the definition of a certain model structure, the Reedy model structure, on the category of simplicial (respectively, cosimplicial) objects in a model category (§3.1). We tackle the construction of the mapping spaces afterwards (in §3.2).

To complete the account of this chapter, we explain the definition of a generalization of the classical geometric realization of simplicial complexes in the setting of model categories and we explain the definition of a generalization of the totalization of cosimplicial spaces (§3.3). This subject is classical and is addressed in reference books on model categories. We just put more emphasis on the possibility to make choices when we determine the geometric realization of a simplicial object in a model category (and when we determine the totalization of a cosimplicial object). To be explicit, we will see that the definition of the geometric realization of a simplicial object depends on the choice of a cosimplicial framing in our model category (and the definition of the totalization dually depends on the choice of a simplicial framing). We precisely check that different choices of cosimplicial frames return homotopy equivalent objects when we pass to the geometric realization (and different choices of simplicial frames return homotopy equivalent totalizations similarly). We rely on a study of homotopy coends (respectively, ends), which we carry out in an appendix section (§3.4), to establish this homotopy invariance result for the geometric realization of simplicial objects in model categories (respectively, for the totalization of cosimplicial objects). By the way, we also survey the definition of general tower decompositions of geometric realizations (and of totalizations).

The content of this chapter is crucial for the subsequent constructions of this monograph. We notably use the tower decompositions of the geometric realization and of the totalization in order to define homotopy spectral sequences which we use to compute the homotopy of homotopy automorphisms spaces of operads in Part III.

Chapter 4. Cofibrantly Generated Model Categories. We review the definition of the notion of a cofibrantly generated model category in this chapter. Briefly say for the moment that a cofibrantly generated model structure enables us to give an effective definition of the class of cofibrations in our model category by using a generalization of the classical notion of a relative cell complex. We explain the definition of this abstract notion of a relative cell complex first (§4.1). We make the axioms of a cofibrantly generated model category explicit afterwards (in §4.2). We also review the applications of the concept of a cofibrantly generated model category to the category of topological spaces and to the category of simplicial sets. We then give an account of the applications of cofibrantly generated model structures to the definition of model structures by adjunction from a base model category (§4.3). We give a brief introduction to the theory of combinatorial model categories, which are
cofibrantly generated model categories equipped with a presentation in the sense of
the classical theory of categories, to complete the account of this chapter (§4.4).
This chapter does not contain any original result and most proofs are abridged or
omitted.

Part II(b). Modules, Algebras, and the Rational Homotopy of Spaces.
We comprehensively revisit the rational homotopy theory of spaces in this part. We
start with a review of the algebraic background of this subject, the homotopy the-
ory of dg-modules, of simplicial modules, and of unitary commutative dg-algebras.
We devote the first and second chapters of the part (§§5-6) to this survey. We rely on the concepts of the theory of model categories recalled in the previous
chapters of this volume. We tackle the applications to the rational homotopy of
spaces afterwards, in the third chapter of the part (§7). We focus on topics which
we subsequently use when we address the definition of our models for the rational
homotopy of operads.

Chapter 5. Differential Graded Modules, Simplicial Modules, and Cosimplicial
Modules. We assume by convention that the objects of the standard category of
dg-modules are equipped with a lower grading (which may run over \(\mathbb{Z}\)) and with
a differential which lowers degree by one. We denote this category of dg-modules
by \(dg\text{-}Mod\). We consider, besides, a category of chain graded dg-modules, which
are equivalent to dg-modules concentrated in non-negative degrees, and a category
of cochain graded dg-modules, which are equivalent to dg-modules concentrated
in non-positive degrees. We adopt the notation \(dg_*\text{-}Mod\) for the category of chain
graded dg-modules and the notation \(dg^*\text{-}Mod\) for the category of cochain graded
dg-modules.

We give a detailed account of the definition of these categories of dg-modules
in the preliminary section of this chapter (§5.0). We also give a brief summary
of the classical Dold–Kan correspondence, the equivalence of categories between
the category of chain graded dg-modules \(dg_*\text{-}Mod\) and the category of simplicial
modules \(s\text{-}Mod\). We formally define the category of simplicial modules \(s\text{-}Mod\) as the
category of simplicial objects in the category of ordinary modules \(\text{Mod}\). We also
deal with the category of cosimplicial modules in our study. We formally define this
category \(c\text{-}Mod\) as the category of cosimplicial objects in the category of ordinary
modules \(\text{Mod}\). We mostly use cochain graded dg-modules when we build our model
for the rational homotopy of spaces. We therefore explain the definition of a model
structure on the category of cochain graded dg-modules with full details (§5.1).

We review the definition of symmetric monoidal structures on the category of
chain graded (respectively, cochain graded) dg-modules and on simplicial (respec-
tively, cosimplicial) modules afterwards. We also recall the definition of (various
forms of) the Eilenberg–Zilber equivalence which we use to relate these symmetric
monoidal categories. We address these subjects in §5.2. We still recall the definition
of internal hom-objects in our categories of dg-modules and in simplicial modules,
and we check the homotopy invariance of these constructions (§5.3).

We devote an appendix section of the chapter (§5.4) to a short review of the
definition of the notion of a contracting homotopy in the context of chain graded
(respectively, cochain graded) dg-modules and of the notion of an extra-degeneracy
(respectively, extra-codegeneracy) in the context of simplicial (respectively, cosim-
plicial) modules.
Most ideas developed in this chapter are not original (apart from the definition of a hom-object counterpart of the Eilenberg–Zilber equivalence in §5.3). Our main purpose is to make explicit the applications of standard constructions of homotopy theory to cochain graded dg-modules after a survey (mostly without proofs) of the homotopy theory of chain graded dg-modules and of simplicial modules.

Chapter 6. Differential Graded Algebras, Simplicial Algebras, and Cosimplicial Algebras. In this chapter, we elaborate on the study of the previous chapter to define a model category structure for unitary commutative algebras. We first make explicit the definition of a unitary commutative algebra in chain graded (respectively, cochain graded) dg-modules by using the symmetric monoidal structure which we attach to this category. We use a similar approach to define the notion of a unitary commutative algebra in simplicial (respectively, cosimplicial) modules. We devote the first section of the chapter to these topics §6.1.

We use the notation $dg^*\text{Com}^+$ (respectively, $dg^*\text{Com}_+$) for the category of unitary commutative algebra in chain graded (respectively, cochain graded) dg-modules. For short, we also call unitary commutative chain (respectively, cochain) dg-algebras the objects of this category of unitary commutative algebras.

We mostly deal with the category of unitary commutative cochain dg-algebras in what follows. We prove that this category inherits a model structure (in the characteristic zero setting) in the second section of the chapter §6.2. We study cell attachments in the category of unitary commutative cochain dg-algebras in-depth in the course of our verifications. We also explain that the homotopy type of a cell attachment of generating cofibrations of unitary commutative cochain dg-algebras can be determined by using a version with coefficients of the bar construction. We devote the third section of the chapter to this subject §§6.2-6.3.

Most statements explained in this chapter are not original (like the constructions of the previous chapter).

Chapter 7. Models for the Rational Homotopy of Spaces. We explain the applications of our model categories of unitary commutative algebras to the definition of models for the rational homotopy of spaces in this chapter. We mainly deal with the Sullivan model which is formed in the category of unitary commutative cochain dg-algebras.

We use a Quillen adjunction to formalize the correspondence between the category of simplicial sets (which we consider instead of topological spaces) and the category of unitary commutative cochain dg-algebras. The Sullivan cochain dg-algebras of piecewise linear forms $\Omega^*(X)$, which is a version of the de Rham cochain complex functor with rational coefficients, gives a (contravariant) functor from simplicial sets to unitary commutative cochain dg-algebras. We recall the definition of this functor $\Omega^* : X \mapsto \Omega^*(X)$ and we make explicit the corresponding left adjoint functor $G_* : A \mapsto G_*(A)$ from the category of unitary commutative cochain dg-algebras to the category of simplicial sets.

In our account, we mainly revisit the proof of the homotopy properties of the Sullivan model, and we give a new interpretation of results of the literature. We notably explain that the homotopy invariance properties of the Sullivan cochain dg-algebra, which we use in the definition of our Quillen adjunction, are related to the definition of a simplicial framing in the category of unitary commutative cochain dg-algebras. We devote the first and the second section of the chapter to these topics (§§7.1-7.2).
The rational homotopy category of spaces can naively be defined as the category which we obtain by formally inverting the maps of spaces that induce an isomorphism on the rationalization of homotopy groups. We devote the third section of the chapter (§7.3) to the study of the correspondence between the homotopy category of the model category of unitary commutative cochain dg-algebras and this rational homotopy category of spaces. We also recall the definition of a rationalization functor on spaces in this section.

These results are well covered by the literature. Therefore we just provide abridged proofs of the statements which we review in this concluding section of the chapter and we refer to the literature for further details.

Part II(c). The (Rational) Homotopy of Operads. In this part, we explain the definition of our models for the rational homotopy of operads. This construction represents the main original theoretical contribution of this monograph.

We start with a detailed study of the definition of model structures on the category of operads in simplicial sets (§8). We then explain the definition of the notion of a (Hopf) cooperad and we check that the category of (Hopf) cooperads in cochain graded dg-modules forms a model category (§9). We define an operadic counterpart of the Sullivan model afterwards (§10) by relying on the definition of this model category of (Hopf) cochain dg-cooperads and by using an operadic upgrade of the Sullivan dg-algebra functor considered in our study of the rational homotopy of spaces.

We actually need to restrict ourselves to (connected) non-unitary operads in our correspondence (in order to handle convergence difficulties with cooperad structures). We can however use an analogue of our notion of a Λ-operad in the category of (Hopf) cooperads in order to extend our model to (connected) unitary operads. We use the same plan as in the case of plain non-unitary operads to carry out the definition of this model for the rational homotopy of unitary operads. We explain the definition of our notion of a (Hopf) Λ-cooperad first (§11) and we check that the category of Hopf Λ-cooperads in cochain graded dg-modules gives a suitable model for the rational homotopy of (connected) unitary operads afterwards (§12).

Chapter 8. The Model Category of Operads in Simplicial Sets. We give a thorough account of the definition of model structures for the category of operads in simplicial sets. We start with a brief inspection of the definition of an operad in simplicial sets (§8.0). We just check that an operad in simplicial sets is equivalent to a simplicial object in the category of operads in sets.

We actually consider two model structures for operads. The first model structure, the one usually given in the literature, will be used in the context of non-unitary operads (operads governing non-unitary algebra structures). The second one, which we introduce in this monograph and call the Reedy model structure, is more appropriate for unitary operads (operads governing algebras with a unit), and will be used in this context. We use our notion of Λ-operad, equivalent to the category of unitary operads, to formalize the definition of this Reedy model structure. We define the model structure of the category of non-unitary operads first (§§8.1-8.2) and the Reedy model structure of the category of Λ-operads afterwards (§§8.3-8.4).

In each case, we use a general adjunction process (recalled in §4.3) to deduce the definition of our model structure on operads from the definition of a model structure
on the category of symmetric sequences (respectively, $\Lambda$-sequences) underlying our objects.

To complete the study of this chapter, we explain the applications of a general construction of simplicial resolutions, the cotriple resolution, for the definition of cofibrant replacements in the category of operads in simplicial sets (§8.5).

Chapter 9. The Homotopy Theory of (Hopf) Cooperads. We explain the general definition of the notion of a cooperad in the setting of a symmetric monoidal category in the first section of this chapter (§9.1). We then explain the definition of a model structure on the category of cooperads in cochain graded dg-modules (§9.2).

We study the category of Hopf cooperads afterwards (§9.3). We formally define a Hopf cooperad in a base category as a cooperad in the category of unitary commutative algebras in this given base category. We check that the category of Hopf cooperads in cochain graded dg-modules inherits a model structure.

We then study the totalization of cosimplicial objects in the category of cochain dg-cooperads and in the category of Hopf cochain dg-cooperads. We mainly prove that the totalization of a cosimplicial object in the category of cochain dg-cooperads can be determined by performing a conormalized cochain construction in the category of cochain graded dg-modules. We devote an appendix section to this subject (§9.4).

Chapter 10. Models for the Rational Homotopy of (Non-unitary) Operads. We define our models for the rational homotopy of non-unitary operads in this chapter. We elaborate on the construction of the Sullivan dg-cochain algebra models for the rational homotopy of spaces. We begin our study with a brief survey of the definition of our model structure for operads in simplicial sets (§10.0). We just check that this model structure admits a restriction to the category of connected (non-unitary) operads, because we have to restrict ourselves to this subcategory of operads in our constructions.

The Sullivan cochain dg-algebra functor $\Omega^*: X \mapsto \Omega^*(X)$ does not preserve multiplicative structures and, as a consequence, does not carry operads to cooperads. This functor preserves multiplicative structures up to homotopy only. The main purpose of this chapter is to explain the definition of an operadic upgrade of the Sullivan functor so that we do can associate a cooperad in unitary commutative cochain dg-algebras (a Hopf cochain dg-cooperad) $\Omega^*_u(P)$ to any operad in simplicial sets $P$. We prove that, under reasonable assumptions on the operad $P$, the components of this Hopf cochain dg-cooperad $\Omega^*_u(P)$ are weakly-equivalent to the Sullivan cochain dg-algebras $\Omega^*(P(r))$ associated to the individual spaces $P(r)$. We use this correspondence to ensure that our construction returns an appropriate result.

We explain these constructions in §10.1. We tackle the applications of our constructions for the definition of our rationalization functor on operads in simplicial sets afterwards, in §10.2.

Chapter 11. The Homotopy Theory of (Hopf) $\Lambda$-cooperads. In this chapter, we study a dual notion, in the category of cooperads, of the category of augmented (connected) non-unitary $\Lambda$-operads which we introduced to model (connected) unitary operads in the first volume of this book. We use the name ‘coaugmented $\Lambda$-cooperad’ for these objects. We also call ‘Hopf $\Lambda$-cooperads’ the objects of the
category of coaugmented \( \Lambda \)-cooperads in any category of unitary commutative algebras. We explain the general definition of a coaugmented \( \Lambda \)-cooperad in the setting of a symmetric monoidal category first (§11.1).

We have an obvious forgetful functor from the category of coaugmented \( \Lambda \)-cooperads to the category of plain cooperads. We check that this functor admits a left adjoint by relying on standard Kan extension constructions (§11.2). We use this correspondence to establish that the category of coaugmented \( \Lambda \)-cooperads in cochain graded dg-modules inherits a model structure (§11.3). We then establish that this model structure lifts to the category of Hopf \( \Lambda \)-cooperads (§11.4).

Chapter 12. Models for the Rational Homotopy of Unitary Operads. In §10, we focus on the study of the rational homotopy of (connected) non-unitary operads. The goal of this chapter is to extend our model to (connected) unitary operads. For this aim, we use the Reedy model structure of the category of augmented \( \Lambda \)-operads in simplicial sets which we defined in §8.

We prove that the functor \( \Omega_* : P \mapsto \Omega_*^\ast(P) \) of §10, which assigns a Hopf cooperad in cochain graded dg-modules to any (connected) operad in simplicial sets \( P \), admits a lifting to the category of (connected) \( \Lambda \)-operads in simplicial sets, and hence, induces a functor between the category of (connected) \( \Lambda \)-operads in simplicial sets and the category of Hopf \( \Lambda \)-cooperads in cochain graded dg-modules.

We begin our study with a brief survey of the definition of our Reedy model structure for \( \Lambda \)-operads in simplicial sets (§12.0). We then explain the definition of our functor \( \Omega_* : P \mapsto \Omega_*^\ast(P) \) on the category of \( \Lambda \)-operads (§12.1) and we explain the applications of this construction to the definition of a rationalization functor on the category of (connected) unitary operads in simplicial sets (§12.2).

Part II(d). Applications of the Rational Homotopy to \( E_n \)-operads.
We make explicit models of the little discs operads to complete our study of the rational homotopy of operads. We precisely check, as we briefly explain in the introduction of this volume, that the Chevalley–Eilenberg cochain complex of (graded analogues of) the Drinfeld–Kohno Lie algebras define such models of the little discs operads in the category of Hopf cochain dg-cooperads. (We have a similar result when we pass to \( \Lambda \)-operads.) We will also explain that this statement can be interpreted as a formality theorem for the little 2-discs operad. Recall that we use the notation \( \hat{\mathfrak{p}} \) for the completion of the ordinary Drinfeld–Kohno Lie algebra operad. Throughout our study, we also use the notation \( \hat{\mathfrak{p}}_n \) for the graded generalization of this Lie algebra operad which we associate to the little \( n \)-discs operad for any \( n \geq 2 \). Thus, we actually have \( \hat{\mathfrak{p}} = \hat{\mathfrak{p}}_2 \).

We recall the definition of the Chevalley–Eilenberg cochain complex of Lie algebras and we make explicit the simplicial sets which correspond to these cochain complexes in the first chapter of the part (§13). We tackle the applications to the little discs operads afterwards (§14).

We recall the definition of a Lie algebra and of the enveloping algebra of a Lie algebra in a preliminary section of this chapter (§13.0). We mainly apply the ideas of the first volume, where we explain a general definition of a Lie algebra in the setting of symmetric monoidal categories, to the base category of dg-modules which we consider in this part. By the way, we review the definition of the notion of a complete filtered module and of a weight graded module in the dg-module context.
We then study the Chevalley–Eilenberg cochain complex of complete Lie algebras in graded modules (§13.1). We prove that the Chevalley–Eilenberg cochain complex of a complete Lie algebra $g$ corresponds, under our model, to a simplicial set of Maurer–Cartan forms $MC_\bullet(g)$ naturally associated to $g$. We explain the definition of a natural decomposition of the Chevalley–Eilenberg cochain complex into a tower of cofibrations in the category of unitary commutative cochain dg-algebras and a parallel decomposition of our simplicial set of Maurer–Cartan forms in the course of our study. We will use a generalization of these tower decompositions to define our homotopy spectral sequence for the computation of the homotopy of the mapping spaces of operads in the next part.

We explained in the first volume that any complete Lie algebra in the ordinary (ungraded) sense $g$ is associated to a Malcev complete group $G$ which we define by taking the group of group-like elements $G = \mathcal{G} \hat{U}(g)$ in the complete enveloping algebra $\hat{U}(g)$ of our complete Lie algebra $g$. We also explained that this Malcev complete group $G = \mathcal{G} \hat{U}(g)$ can be depicted as a group of exponential elements $e^\xi$, $\xi \in g$, in the complete enveloping algebra $\hat{U}(g)$.

We actually have a weak-equivalence between the simplicial set of Maurer–Cartan forms $MC_\bullet(g)$ which we associate to the Chevalley–Eilenberg cochain complex of the Lie algebra $g$ in this chapter and the classifying space $B(G)$ of the Malcev complete group $G = \mathcal{G} \hat{U}(g)$. We explain this relationship in the concluding section of the chapter (§13.2). We also make explicit the definition of an analogue, for the classifying space $B(G)$ of the group $G = \mathcal{G} \hat{U}(g)$, of the tower decomposition of the simplicial set of Maurer–Cartan forms $MC_\bullet(g)$.

This chapter does not contain any original result. We mainly revisit classical constructions in our framework for the applications to operads of the next chapter.

Chapter 14. Formality and Rational Models of $E_n$-operads. In this chapter, we study the applications of the Chevalley–Eilenberg cochain complex to operads in Lie algebras. We aim to make explicit the models of $E_n$-operads in our category of Hopf dg-cooperads. We also explain the definition of a natural tower decomposition of these Hopf dg-cooperad models of $E_n$-operads. In fact, we deal with Hopf Λ-cooperads rather than ordinary Hopf cooperads in our constructions. We therefore study models of $E_n$-operads in the category of Hopf Λ-cooperads in cochain graded dg-modules. We first study an additive version of the notion of a Hopf Λ-cooperad which naturally occurs when we consider the fibers (actually, the cofiber) of these tower decompositions.

We explain this concept in a preliminary section of the chapter (§14.0). We study the Chevalley–Eilenberg cochain complex of (graded generalizations of) the Drinfeld–Kohno Lie algebra operad $\hat{p}_n$ afterwards (§14.1). We check that the Chevalley–Eilenberg cochain complex $C_{CE}^* (\hat{p}_n)$ of this operad in the category of complete chain graded Lie algebras $\hat{p}_n$ forms a cofibrant object in the category of Hopf cochain dg-Λ-cooperads, for any $n \geq 2$. We explain, by the way, that the correspondence between the Chevalley–Eilenberg cochain complex and the simplicial sets of Maurer–Cartan forms studied in the previous chapter extends to the category of operads. We also study the applications of the tower decompositions of the previous chapter to the Chevalley–Eilenberg cochain complex $C_{CE}^* (\hat{p}_n)$ of the graded Drinfeld–Kohno Lie algebra operad $\hat{p}_n$ and to the corresponding spaces of Maurer–Cartan forms. We then explain the statement of formality results for...
$E_n$-operads which imply that the object $C^*_{CE}(\hat{p}_n)$ is weakly-equivalent to our model $\Omega^*_{\delta}(D_n)$ of the little $n$-discs operad $D_n$.

We devote the next section of the chapter (§14.2) to the particular case $n=2$ of the study of the little discs operads. We then consider the operad of chord diagrams $CD^\vee$, which consists of the Malcev complete groups $CD(r)^\vee = C(\hat{U}(\hat{p}(r)))$, associated to the standard (ungraded) complete Drinfeld–Kohno Lie algebras $\hat{p}(r) = \hat{p}_2(r)$, $r > 0$. We prove that the classifying space of these Malcev complete groups define an operad in simplicial sets $B(CD^\vee)$ which is weakly-equivalent to the rationalization $D_2^\vee$ (in our sense) of the little $n$-discs operad $D_2$. We rely on the existence of Drinfeld’s associators, of which we explained the definition with full details in the first volume of this monograph, to establish this result. We also examine the applications, to this operad $B(CD^\vee)$, of the tower decomposition of the classifying spaces of Malcev complete groups.

We give a short reminder on the definition of the Drinfeld–Kohno Lie algebra operad in an appendix section (§14.3) to complete the account of this chapter.

**Part III. The Computation of Homotopy Automorphism Spaces of Operads.** We complete the computation of the homotopy of the homotopy automorphism space of the rationalization of $E_2$-operads in this part. We explain a general method of computation of the homotopy of mapping spaces of operads in a first step. We tackle the applications of this method to $E_n$-operads and to $E_2$-operads afterwards.

**Introduction to the Results of the Computations for $E_2$-operads.** We first recall the statement of our main theorem, the identity between the (pro-unipotent) Grothendieck–Teichmüller group and the group of homotopy automorphism classes of the rationalization of $E_2$-operads, which represents the main objective of this part, and we explain the plan of our computation method.

**Part III(a). The Applications of Homotopy Spectral Sequences.** Recall that the homotopy automorphism space of an object $X$ in a model category $\mathcal{C}$ consists of the invertible connected components of the mapping space with $X$ as source and target object. We use homotopy spectral sequences to compute the homotopy of such mapping spaces in the context of operads. We explain the general definition of these homotopy spectral sequences first (§1). We prove that, in the context of operads, the second page of our homotopy spectral sequences has a conceptual description in terms of a natural cohomology theory, the cotriple cohomology, which we define on the category of operads in graded modules (§2). We explain a general computation method of this cohomology of operads, by using duality theories, namely the bar duality and the Koszul duality of operads, which give small resolutions of operads (§3).

**Chapter 1. Homotopy Spectral Sequences and Mapping Spaces of Operads.** The homotopy spectral sequences, which we use in our computations, have actually been defined by Bousfield-Kan, and we give a short survey of the general definition of these spectral sequences (mostly without proofs) before tackling the applications to mapping spaces of operads.

We first briefly explain our terminological conventions for spectral sequences (§1.0). Let us mention that the homotopy spectral sequences are generally formed in the category of sets and some care is necessary in this context. We then recall the definition of a homotopy spectral sequence associated to a simplicial set equipped
with a decomposition into the limit of a tower of fibrations and the definition of a homotopy spectral sequence associated to the totalization of a cosimplicial space (§1.1).

We use both constructions when we deal with mapping spaces of operads. We actually take objects which naturally occur as a limit of a natural tower of fibrations in the category of operads on the target of our mapping spaces. (We deduce such decompositions from the study of the previous part in the case of $E_n$-operads.) This tower decomposition in the category of operads gives a decomposition as the limit of a tower of fibrations at the mapping space level. We consider, on the other hand, the geometric realization of a simplicial resolution (the cotriple resolution) as a source object in our mapping spaces. This construction implies that our mapping spaces occur as the totalization of cosimplicial spaces and we apply the homotopy spectral sequence of cosimplicial spaces to such objects. We explain these ideas in the concluding section of the chapter (§1.2).

Let us observe that we actually get a double spectral sequence for our mapping spaces, with a horizontal spectral sequence direction which arises from the simplicial decomposition of the source object and a vertical spectral sequence direction which arises from the tower decomposition of our target object. We mostly give methods to compute the horizontal (cosimplicial) homotopy spectral sequence in the next chapters of this part.

Chapter 2. Applications of the Cotriple Cohomology of Operads. We first explain that the terms on the second page of this operadic cosimplicial homotopy spectral sequence reduces to the cotriple cohomology of the homology of our operads. This result gives the starting point of our subsequent computations.

We heavily use multi-graded structures in our study of spectral sequences and we devote a preliminary section to a detailed survey of this background (§2.0). We then explain the definition of a category of abelian bimodules over operads, which give the general notion of coefficients which we associate to the cotriple cohomology of operads (§2.1). We explain the definition of the cotriple cohomology itself afterwards. We explicitly define the cotriple cohomology $H^\ast_{\Lambda,\op}\Lambda_{\op}(R, N)$ of an operad in graded modules $R$ with coefficients in an abelian bimodule $N$ as the cohomology of a cosimplicial module of operadic derivations $\Der_{\Lambda,\op}(R, N)$ defined on the cotriple resolution $R_\ast = \Res_\ast(R)$ of our operad $P$ and with values in our abelian bimodule $N$ (see §2.2). For our purpose, we just focus on the applications to augmented $\Lambda$-operads rather than to ordinary operads (as our notation indicates). We quickly check that this cotriple cohomology theory does determine the second page of our operadic cosimplicial homotopy spectral sequence (as we expect).

We devote an appendix section of this chapter to a thorough study of the homotopy properties of hom-objects on the categories of symmetric sequences and $\Lambda$-sequences which underlie our operads (§2.3).

Chapter 3. Applications of the Koszul Duality of Operads. We then explain the applications of the bar duality and of the Koszul duality of operads to the definition of reductions of the cotriple cohomology complex which we introduced in the previous chapter. We carry out this reduction process itself in the first section of this chapter (§3.1). To complete this study, we just review the definition of our derivation complex in order to make explicit the structure of the reduced complexes which we associate to the bar construction and to the Koszul construction of operads. We address these topics in the second section of the chapter (§3.2).
Part III(b). The Case of $E_n$-operads. In this part, we examine the application of the Koszul construction, which represents the final outcome of the study of the previous chapter, to the homology of the little discs operads (equivalently, of $E_n$-operads). We eventually completely determine the second page of our operadic cosimplicial homotopy spectral sequence in the case of $E_2$-operads (§4). Then we check that the classes which we obtain in this homotopy spectral sequence computation correspond to a natural decomposition of the Grothendieck–Teichmüller group to complete the verification of the main result of this work, the homotopy interpretation of the pro-unipotent Grothendieck–Teichmüller group as the group of homotopy automorphisms of the rationalization of $E_2$-operads (§5).

Chapter 4. The Applications of the Koszul Duality for $E_n$-operads. We recalled in the first volume of this monograph that the homology of the little $n$-discs operad is identified with a graded version of the operad of Poisson algebras. In this book, we also use the notation $\text{Gerst}_n$ and the name ‘$n$-Gerstenhaber operad’ for this operad such that $\text{Gerst}_n = H_*(D_n)$, because the 2-Gerstenhaber operad, for which we also use the simplified notation $\text{Gerst} = \text{Gerst}_2$, is identified with the operad that governs the kind of algebra structures introduced by Gerstenhaber for the study of the deformation complex of algebras.

The $n$-Gerstenhaber operad $\text{Gerst}_n$ is an instance of a Koszul operad. We recall the statement of this Koszul duality result in the first section of this chapter (§4.1) and we determine the cohomology of the associated derivation complex afterwards (§4.2).

Chapter 5. The Interpretation of the Result of the Spectral Sequence in the Case of $E_2$-operads. The main consequence of the result of the previous chapter is that the cotriple cohomology of the 2-Gerstenhaber operad, and hence, the second page of the operadic cosimplicial homotopy spectral sequence for $E_2$-operads, is identified with the graded Grothendieck–Teichmüller Lie algebra $\text{grt}$. In this chapter, we review the definition of our mapping from the pro-unipotent Grothendieck–Teichmüller group $GT(Q)$ to the space of homotopy automorphisms of the rationalization of $E_2$-operads $\text{Aut}^h_{O_p^*}(E_2^\wedge)$ (§5.0). We check that our identity at the spectral sequence level reflects a natural tower decomposition of this group $GT(Q)$. We deduce from these observations that our double spectral sequence degenerates and that our mapping gives a bijection $GT(Q) \cong \pi_0 \text{Aut}^h_{O_p^*}(E_2^\wedge)$ when we pass to the degree zero homotopy of our homotopy automorphism space $\text{Aut}^h_{O_p^*}(E_2)$ (§5.1). We also rely on our spectral sequence computations to check that the homotopy of this homotopy automorphism space reduces to a module of rank one in degree one and vanishes in degrees larger than one (§5.2). This verification completes the proof of our main statement, such announced in the foreword of the first volume.

We just devote an appendix section of the chapter to the verification of a (partial) idempotence property of the rationalization of $E_2$-operads (§5.3). We mainly use the observation of this appendix section to give a simple interpretation of our result.

Conclusion: A Survey of Further Research on Operadic Mapping Spaces and their Applications. To conclude this study, we outline new developments of the homotopy theory of $E_n$-operads. Notably, we give a brief statement of the generalization of the computation of the previous chapter for the homotopy automorphism spaces of the rationalization of $E_n$-operads. These computations, carried out by the author in a joint work with Victor Turchin and Thomas
Willwacher, heavily rely on graph complexes similar to the graph complexes introduced by Kontsevich at the origin of the renewal of the theory of operads in the 1990’s. We also briefly explain the applications of mapping spaces of $E_n$-operads to the study of the embedding spaces of euclidean spaces mentioned in the introduction of this volume.

Appendix C. Cofree Cooperads and the Bar Duality of Operads. In this appendix, we first examine a dualization of the constructions of §§A-B with the aim of giving an explicit definition of cofree objects in the category of cooperads. We briefly recall our conventions on trees first (§C.0). We tackle the construction of cofree cooperads afterwards (§C.1). We then survey the ideas of the bar duality and of the Koszul duality of operads (§§C.2-C.3).

Most concepts which we explain in this appendix chapter are not original. We mainly survey definitions of the literature and we generally skip the proof of theorems. We just check that the standard constructions extend to our $\Lambda$-cooperad (and $\Lambda$-operad) setting. (We will see that this extension enables us to apply the bar duality and the Koszul duality theory to unitary operads.)