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References to chapters, sections, paragraphs, and statements of the book are given by §x.y.z when these cross references are done within a part (I, II, and III), and by §P.x.y.z where P = I, II, III otherwise. The cross references to the sections, paragraphs, and statements of the appendices are given by §P.x.y all along the book, where §P = §A, §B, §C. The preliminary part of the first volume of this book also includes a “foundations and conventions” section, whose paragraphs, numbered §§0.1-0.16, give a summary of the main conventions used in this work.

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