Preface

Many problems in measure theory, probability theory, and diverse applications are connected with various types of convergence of measures. The most frequently encountered is weak convergence, but often one has to deal with other modes of convergence, for example, in variation or setwise. In the form of convergence of distribution functions, weak convergence of measures appeared actually at the dawn of probability theory, and now it has become one of the most important tools in applied and theoretic statistics. Many of the key results in probability theory and mathematical statistics can be regarded as statements about weak convergence of probability distributions. The foundations of the theory of weak convergence of measures were laid by J. Radon, E. Helly, P. Lévy, S. Banach, A.N. Kolmogorov, V.I. Glivenko, N.N. Bogoliubov, N.M. Krylov, and other classics from the 1910s through the 1930s. The formation of this theory as a separate field at the junction of measure theory, probability theory, functional analysis, and general topology is connected with fundamental works of A.D. Alexandroff at the end of the 1930s and the beginning of the 1940s, and this theory gained its modern form after the appearance of the outstanding paper of Yu.V. Prohorov in 1956. An extremely important role was also played by the book by B.V. Gnedenko and A.N. Kolmogorov on limit theorems of probability theory and the works of L.V. Kantorovich on optimal transportation. More details are given in the comments.

Convergence of measures is the subject of a vast literature (see the comments), in particular, weak convergence of measures is discussed in detail in the author’s two-volume book *Measure theory* (see [81]). However, already at the time of working on that book, it was clear that convergence of measures deserved a separate exposition, which was impossible in a book of broad thematic coverage such as [81]. In spite of the fact that all principal results related to convergence of measures are fully presented in Chapter 8 of [81], such a presentation cannot be qualified as exhaustive and sufficient for a broad readership. First of all, the presentation in [81] is oriented towards experienced readers and, by the necessity of keeping the size of the book within reasonable limits, is rather condensed. Secondly, due to the same constraint on book size, justifications of many interesting results and examples there were delegated to exercises, and although they contained hints, they were even more condensed. Finally, the discussion of applications in [81] is reduced to a minimum. The goal of this new book is a more accessible and paced presentation of the theory of weak convergence of measures and some other important types of convergence, oriented towards a broad circle of readers with different backgrounds. Certainly, the subject itself unavoidably presupposes certain minimum of prerequisites (presented in the first chapter), but the material is organized in a form which attempts to postpone for as long as possible the employment of any specialized knowledge. In this respect I followed the example of Billingsley, the author of a
beautiful introductory book [67] on weak convergence of probability measures (I began my acquaintance with this subject using this book many years ago); though, unlike his text, this book includes considerably less elementary material for advanced readers. Thus, here we offer two levels of presentation: rather elementary material in the main sections of Chapters 1–3, and some more specialized information presented in the complements to all chapters and also in Chapters 4 and 5. Such a structure leads to the effect that some concepts and results appear first in relation to measures on the real line or on $\mathbb{R}^d$, next when considering measures on metric spaces, and finally in the general case of topological spaces. In this way the book combines features of a textbook and an advanced survey. Certainly, the material of the aforementioned Chapter 8 of [81] is completely covered by this book, but most of the proofs from that chapter have been reworked: following suggestions and corrections received from my readers, more details have been added and many gaps and inaccuracies have been corrected. In addition, a number of interesting results given in [81] only with formulations are now supplied with complete justifications. Although a number of classical principal results are given with exactly the same formulations as in [81] and slightly revised proofs (such examples can be found, e.g., in Sections 2.3, 2.6, 3.1, 4.2, 4.3, and 5.1), in many other cases the formulations have been altered as well. This is not because the old formulations were not satisfactory, but rather because the whole structure of the text has been changed significantly. In particular, the case of metric spaces is now studied first and does not come as a special case of the general situation as in [81]. Many relatively old and some very recent results included in the book have also contributed to its size being nearly three times more than that of Chapter 8 of [81]. Certainly, the bibliography has been considerably updated: More than 100 works in the references have been published over the last decade, and this is a small portion of the available literature. In particular, many authors presented in this bibliography have much longer lists of related publications so that I had to be very selective when preparing the bibliography.

In Chapter 1, after presenting some necessary facts from integration theory and functional analysis in the first three sections, we discuss the simplest notions and facts related to convergence of measures on the interval and the real line and also on $\mathbb{R}^d$. However, even these basic concepts are useful for a very broad circle of problems that ever touch on anything related to convergence in distribution and weak convergence of measures. Phenomena discussed here illuminate well the general situation. Specific for the one-dimensional case is analysis of convergence of distribution functions. In this chapter we also study the Fourier transform (the characteristic functionals).

In Chapter 2, still at a rather elementary level, the discussion moves to metric spaces, but here some topological concepts already show up. The central results of this chapter are connected with the theorem of Yu.V. Prohorov on weak compactness, the theorem of A.D. Alexandroff on convergence of probability measures, and the parametrization of weakly converging measures due to A.V. Skorohod. Separate sections or subsections are devoted to weak convergence of measures on various special spaces such as Hilbert, Banach or some concrete functional spaces. In this chapter weak convergence is considered not only for countable sequences, but also for more general uncountable nets. Mostly, this does not lead to any complications, but is useful from the point of view of general ideas (especially with a view towards
the continuation of our discussion for topological spaces). However, in all places in Chapter 2 (but not in Chapters 4 and 5) where nets are mentioned, it is quite possible to assume that these are usual sequences.

In Chapter 3 we consider metrics on spaces of measures (in particular, we discuss the Prohorov, Kantorovich, Kantorovich–Rubinshtein, and Fortet–Mourier metrics), and we also give a brief introduction to the theory of Gromov metric triples. Separate subsections are devoted to Zolotarev metrics and certain special questions connected with various estimates.

In the past two decades this area has been intensively developing in close connection with another very popular modern direction—optimal transportation. However, this very important aspect is not touched on in the present book because any sufficiently detailed discussion would considerably increase the size of the text.

A more advanced exposition requiring some knowledge of basics of general topology and some experience of working with topological spaces starts in Chapter 4 and ends with a discussion of topological properties of spaces of measures in Chapter 5. Chapter 4 begins with a brief exposition of fundamentals of measure theory in general topological spaces, then the weak topology on spaces of measures on general spaces is discussed including A.D. Alexandroff’s results in this general setting. Among other things, compactness in the weak topology is thoroughly studied. We return to Prohorov’s theorem in this framework, which leads to an interesting class of topological spaces, the so-called Prohorov spaces. Fourier transforms of measures on locally convex spaces are introduced and considered in relation to weak convergence. These themes are continued in Chapter 5, where the main emphasis is on topological properties of spaces of measures equipped with the weak topology. Here we also return to Skorohod representations. Separate sections are devoted to setwise convergence topology and the $ws$-topology, which is a mixture of the weak and setwise convergence topologies. Both have interesting connections with our main subject. Uniformly distributed sequences in topological spaces is another related topic discussed in this chapter.

Each chapter ends with a collection of exercises including easy exercises and more subtle facts (with hints or references to the literature; some of such advanced exercises are in fact very difficult and, in principle, could be placed as theorems in the text with references to their sources, but their inclusion in the form of exercises may be regarded as an invitation to seek simpler solutions).

The book ends with brief historic and bibliographic comments, a list of references (with indications of all pages where they are cited), and the subject index (which begins with a list of notations).

For reading this book it is useful, although not necessary at all, to be acquainted with basics of probability theory, the problems, ideas, and methods of which are of great importance for the area we discuss. In addition to the known fundamental treatises, including Ash [24], Bauer [44], Billingsley [66], Borovkov [108], Chow, Teicher [138], Cramér [146], Dudley [193], Feller [221], Fristedt, Gray [246], Gänssler, Stute [254], Gnedenko [281], Hennequin, Tortrat [318], Hoffmann-Jørgensen [328], Kallenberg [343], Loève [437], Neveu [482], Rotar [556], Shiryaev [581], and Tortrat [617], I would note an elegant introduction by Lamperti [408].

A considerable part of the material in this book was presented by the author in lectures at the Department of Mechanics and Mathematics of Moscow State University, at the Independent Moscow University, at the Faculty of Mathematics
of the Higher School of Economics in Moscow, and also in lectures and talks at other universities and mathematical institutes all over the world, including the Steklov Mathematical Institute of the Russian Academy of Science in Moscow and its St. Petersburg Department, Kiev, Berlin, Bonn, Bielefeld, Paris, Strasbourg, London, Cambridge, Warwick, Rome, Pisa, Copenhagen, Stockholm, Delft, Vienna, Barcelona, Lisbon, Athens, Berkeley, Boston, Minneapolis, Vancouver, Montreal, Edmonton, Haifa, Tokyo, Kyoto, Beijing, Sydney, and Santiago.


The book also includes results obtained in research supported by the Russian Science Foundation (Grant 17-11-01058 at Lomonosov Moscow State University).

Moscow, Russia
Spring 2018