Preface

Overview

The vibrant interaction between function theory and functional analysis is more than one hundred years old \cite{262}, \cite{277}, and it continues to prosper. In this volume we consider topics in that tradition, all centered around the classical Dirichlet space, $\mathcal{D}$. That space is the set of functions $f$ which are holomorphic on the disk having finite Dirichlet integral:

$$\mathcal{D}(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'|^2 \, dA < \infty.$$ 

In the first part of the book we focus on $\mathcal{D}$ as a reproducing kernel Hilbert space. From that viewpoint we investigate multipliers, Carleson measures, zero sets, and interpolating sequences for both the space and for its multiplier algebra. The fact that the reproducing kernel for $\mathcal{D}$ has the complete Pick property is often a central factor.

In the second part of the book we consider more specialized topics related to $\mathcal{D}$, including discussions of interpolation, boundary behavior, alternative norms, the local Dirichlet integral, shift invariant subspaces, and Hankel forms. These treatments are largely independent of each other and can be read in any order. In those sections the Hilbert space structure of $\mathcal{D}$ is still used, but the general theory of reproducing kernel Hilbert spaces has a smaller role and the complete Pick property is rarely mentioned. On the other hand estimates from function theory and potential theory have a larger role.

Some of the results we discuss are in a definitive form. Others are only incomplete resolutions of natural questions. In fact, for many interesting questions about the Dirichlet space, including describing the zero sets, the boundary zero sets, the onto interpolating sequences, and the shift invariant subspaces, only partial results are known.

In the final part we expand our viewpoint in two fundamental ways. We move beyond functions of a single variable and consider spaces of holomorphic functions on the ball in $\mathbb{C}^n$. Also, we go beyond Hilbert spaces and consider Banach spaces of functions. With these changes, new techniques need to be introduced. In particular, analysis of the Besov spaces on the ball requires obtaining good local estimates for higher order derivatives in the context of the anisotropic geometry of the ball, and doing so requires new machinery. Also, as we move to Banach spaces of functions we lose access to tools from the theory of reproducing kernel Hilbert spaces. As a consequence, for instance, when characterizing interpolating sequences in the ball, we obtain interpolating functions constructively, not by abstract existence theorems as done in one variable. Those chapters present basic facts about Carleson measures,
multipliers, and interpolating sequences for spaces of holomorphic functions as well as for their discrete model spaces.

The final chapter has a corona type theorem for Besov spaces on the ball as well as a classical multiplier corona theorem for the Hilbert spaces in the family, including $H^2$, $D$, and the finite dimensional Drury-Arveson spaces. Interestingly, at this point the discussion comes full circle. The theory of reproducing kernel Hilbert spaces with complete Pick kernels, a tool that had been set aside, is used for passage from “corona type theorems” to classical “corona theorems.”

The final chapters contain only a few theorems, some with elaborate proofs. However we feel, and hope some readers feel, that the true heart of those sections lies beyond those specific results, and is in the techniques and tools that are developed. Those techniques include a tool for quantifying local oscillation of Besov space functions, the “almost invariant holomorphic derivative”, the development of new types of function spaces on trees which model spaces of holomorphic functions on the ball, and new methods for solving $\partial$ equations in several variables with estimates in Besov spaces.

There are two themes which run as subtexts throughout the book. The first is the background role of the Hardy space $H^2$. The study of $H^2$ is the historical and conceptual basis of much of modern function theoretic operator theory and understanding how classical Hardy space results do or do not generalize is a continuing major theme. There is also the more recent insight that both $H^2$ and $D$ are Hilbert spaces whose reproducing kernels have the complete Pick property and, as such, they share some basic properties. We will discuss the commonalities as we go along.

The other ongoing theme is the study of discrete “tree” models of the disk and ball, as well as function spaces on those trees. The simplest of those are, in the language of physicists, toy models. It is the authors’ experience that spaces on trees are very valuable tools, both conceptually and technically. In particular, they are used in essential ways in the proofs of Theorems 13.2, 15.1, and 15.2. Also, it is the authors’ opinion that these models are intrinsically very interesting. As we go along we present those views in detail.

The book includes two appendices. The first is a small collection of results from functional analysis. The second concerns Schur’s lemma, a method for proving boundedness of linear operators given by positive kernels. We use that basic tool in several contexts and with several variations, and we felt it would be convenient for the authors, and we hope for the readers, to collect that material in a single location.

Most material is presented at the level of a topics course, appropriate for students who have taken the introductory graduate course (e.g. complex analysis, measure theory, and functional analysis). Selections from the material in the first seven chapters could be used as a basis for a graduate course at the level just beyond the qualifying exams. Various subsets of the later chapters could be used as the basis for an advanced graduate course, reading course, or seminar. The material in those later chapters is generally more sophisticated; in particular the discussion of “holomorphic Besov spaces on trees” and the proof of the corona theorem for the Drury-Arveson space are best suited for advanced students or researchers.

Although much of our presentation is self contained, some parts are not. We try to remain clear as we go along which mode we are in, and ask the readers indulgence
for the times we are unclear. In the early chapters, working with functions of one
variable, we do not offer full discussions of Hardy space theory or of potential theory
on function spaces. In later chapters, working with functions of several variables,
we use external sources for basic facts about the geometry of the complex ball, and
about the functional analysis of Besov spaces. In the final chapter, working on
corona questions, we make use of relatively recent research on solution operators
for $\partial$ equations and on the Koszul complex.

Also, as we go along we will mention without proof some interesting results
related to our main narrative; sometimes with a small discussion, in other cases
with just a reference.

There are a number of exercises in the first part of the book. Their intention
is to encourage the reader to keep the pencil and paper nearby:

Can one learn mathematics by reading it? I am inclined to say no.
Reading has an edge over listening because reading is more active
– but not much. Reading with pencil and paper on the side is very
much better – it is a big step in the right direction. The very best
way to read a book, however, with, to be sure, pencil and paper
on the side, is to keep the pencil busy on the paper and throw the
book away.

Paul Halmos, “The problem of learning to teach”, American Mathematical Monthly 82
(1975), 466–476.

These are rapidly evolving topics. Our hopes of providing an up-to-date and
comprehensive treatment collapsed under the weight of material. Thus, this is not
a systematic survey of the areas mentioned; rather we present questions, answers,
and techniques that the authors hope are an invitation to the area. We begin each
chapter with a brief discussion to help create a context for that chapter. We focus
on the specifics we know best and which we hope move our narrative forward. We
end chapters with a section of notes which include some supplemental references
and comments.

We are telling a very rich story, and our telling reflects both our interests and
our ignorance. Our apologies to those who are ill treated in our telling.

Sources

The theory of the Hardy space, which is the background arena for many of the
early topics, is presented in many places. The sources the authors rely on heavily
include [161] and [245].

The theory of the Dirichlet space is presented in [136] and some of that material
is also presented here. That presentation gives more emphasis to the potential
theoretic aspects of the theory, ours to the Hilbert space theoretic. There are
also surveys on the Dirichlet space which discuss many of the results here; see the
following [340], [287], [51], [205].

Our discussion of interpolating sequences is informed by the presentations in
[161], [245], [302], [294], and [4].

The book [4] is a basic reference for Hilbert spaces with reproducing kernels
as well as the more specialized theory of kernels with the complete Pick property.
The more recent [255] also contains a great deal of related material.

The books [288], [346], and [345] are our basic references for function theory
and function spaces on the unit ball in complex Euclidean space.
Background about corona problems in several variables is in [130], [129], and [196].

A substantial fraction of the contents here is from research papers, new and old, by the authors and others. We will mention those as we go along; in the text, in the notes at the end of each chapter, or both. However we will certainly overlook some relevant references and we apologize for that.

Some of the material we discuss has not been published before. That includes material in the widely circulated but unpublished manuscripts [3], [72], and [220]; as well as from the more recent [47]. Some results here are new; including the John-Nirenberg type theorem in Section 3.8, and the results on interpolating sequences for Besov spaces on the ball, beginning in Chapter 14, which completes the program begun in [47].

There are also informal presentations available of some of the topics here. Although we do not use them as references in the text, readers may find them useful. Evolved versions of some of that material is in this volume. First, in the academic year 2010-2011 Brett Wick ran an Internet Analysis Seminar on the topic “The Dirichlet Space.” The lectures and some other material from that seminar are available online [339]. Second, in the summer of 2011 the Mathematical Sciences Research Institute sponsored a Summer Graduate School titled “The Dirichlet Space, Connections Between Operator Theory, Function Theory, and Complex Analysis.” Videos of many of the lectures given there, as well as other material, is available online at [221]. Finally, early versions of the first part of the book [255] are, as of now, available online [254].

Finally, the general viewpoint here as well as some details are similar to those in [294], generally intentionally.

We have additionally some papers that are tangential to the themes of this book; while they are not immediately connected the interested reader can consult them and see the commonalities between this book and tools and techniques in the papers [14, 61, 81, 82, 100, 137, 138, 140, 151, 153, 167, 169, 206, 211, 233, 244, 253, 286, 310, 319, 321, 322, 349].

Topics Not Included

As often happens in presentations of mathematical material, we begin by noting some things we will not talk about. Here are a few.

For many mathematicians “Dirichlet spaces” is a topic in abstract potential theory which traces back to the classical work of Beurling and Deny [71]. Although $D$ is their basic model, that work extends the ideas in directions that are very different from those we consider. We will mention it no more.

We will introduce trees and function spaces on trees. That is a huge topic and we will not even try give an overview of it. The history goes back, at least, to Poincaré [162]. We only present our small corner of that larger picture. Some samples of work that resonates with what we are doing here are [195], [261], [216], [204].

Also, we do not give a systematic development of the potential theory associated with spaces of smooth functions, or even the details of potential theory associated to the Dirichlet space. Sources for that material include [1] and [266] as well as the discussions in [136]. Nor do we systematically consider potential theory for function spaces on trees; see [216] and the references there.
Although tempted, we do not have a discussion of corona questions for spaces of functions on the disk.

**Bookkeeping and Notation**

We take an inconsistent policy on the normalizing constants associated with integrals. Often, when we know them we include them. However when we are unsure, or when we feel the exact value is not essential, we use a generic “$C$” or “$c$” to denote that unspecified constant. Thus, the letter is an oxymoronically named *variable constant*. Worse still, at times we will forget to include the constant.

We will generally use the notation $A(x) \lesssim B(x)$ to indicate that there is an unspecified, but finite, $C$ such that, for all $x$ (or all $x$ in an obvious range) $A(x) \leq CB(x)$. This avoids introducing $C$ as another variable constant. We write $A(x) \approx B(x)$ if both $A(x) \lesssim B(x)$ and $A(x) \gtrsim B(x)$ hold. We will write $f \sim g$ when we mean that there is a formal relationship between $f$ and $g$ that will be useful as a guide to the reader.

All our Hilbert spaces are assumed separable.

We will often use the notation $A := B$ to indicate that the expression on the left is being defined by the expression on the right.

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