This book is an introduction to Euclidean and non-Euclidean geometry designed for an upper-level college geometry course. Its content and narrative grew out of our experience teaching junior/senior level advanced geometry and history of mathematics courses for over twenty years. We have learned that, independent of ability and mathematical maturity, most students have only faint memories of the synthetic Euclidean geometry that they studied in high school. This, coupled with the fact that mathematics students rarely have occasion to read primary sources in their major, led us to take full advantage of the unique opportunity available in geometry and introduce our readers to the bible of mathematics, Euclid’s *Elements*. With Euclid as a guide, the reader begins by travelling along the same path as millions of geometry students spanning multiple millennia, continents and languages.

Before beginning our journey with Euclid, we introduce the two most important and familiar geometric objects as our main characters, *the line* and *the circle*. These characters coincide with the Euclidean tools provided by the axioms at the start of the *Elements*. For us, they serve as a narrative touchstone as we periodically check in on them as we make our way through the *Elements*, noting that, while some books emphasize the line, others highlight the circle. In Book I, for example, the line has the starring role in the proposition statements, but the circle is doing a lot of the heavy lifting behind the scenes within the proofs. When we move beyond Euclidean geometry, we identify the behavior of our main characters in each of their new environments in order to keep track of any changes. Comparing and contrasting the nature of these fundamental figures in other geometries has the added benefit of challenging the reader’s preconceived notions of straightness and roundness, forcing a re-examination of basic geometry concepts. In particular, we encourage the reader to contemplate provocative questions such as: What is a straight line? What does parallel mean? What is distance? What is area?

Euclid’s *Elements* is a mathematical achievement of historical significance. No other mathematics text has been published as many times or read by as many people as the *Elements*. Its longevity is due to its clarity, rigor and, most importantly, its superior organization and development of geometry as an axiomatic system. It is the proper gateway to the study of axiomatic systems. Of course, the only way to fully appreciate and understand Euclidean geometry is to step outside of it in order to gain perspective. We do this by exploring Spherical, Taxicab, Hyperbolic, and finite affine and projective geometries. In fact, we take a detour halfway through Euclid’s first book in order to consider two of these geometries. This change of perspective at an early stage of our exploration of the *Elements* provides a natural way to expose hidden flaws in Euclid’s reasoning. It also sheds light on the importance of the axiomatic development.
of mathematics, and creates an avenue to discuss the difference between axiomatic systems and their models, as well as the desirable properties of such systems.

The history of geometry comprises the majority of the history of mathematics. Fittingly, this text includes discussions of many important historical figures and developments in geometry, including the controversy surrounding Euclid’s fifth postulate, the impossible constructions of Greek antiquity, and the development of non-Euclidean, projective and finite geometries. It is important for students to understand that new mathematics does not arrive fully formed on the page, but rather, evolves as it is discovered and created by individuals, sometimes spanning centuries.

Outline of the book

This book covers traditional Euclidean geometry with an axiomatic approach through the lens provided by the *Elements*. After introducing our two main characters in Chapter 1, we discuss Euclid’s definitions, Postulates, and Common Notions in Chapter 2. Chapter 3 presents Neutral geometry, covering the first 28 propositions of Book I along with I.31. We take a detour from the *Elements* to explore Spherical and Taxicab geometries in Chapters 4 and 5. These geometries force us to reconsider our preconceived ideas about straightness, distance and area. They also reveal the gaps in Euclid’s original set of axioms. Opening the door to other geometries early in the book allows us to consider axiomatic systems in general, and to introduce Hilbert’s axioms for plane geometry in Chapter 6 as a way to shore up the gaps in Euclid’s foundation. Here, we also detail the fundamental and desirable properties of axiomatic systems and the mathematicians who were the first to successfully navigate these metamathematical waters.

After discussing Neutral, Spherical and Taxicab geometries, we head back to Euclid’s Book I in Chapter 7. By including Book I in its entirety, readers experience the beauty of Euclid’s reasoning and his modular approach to the systematic development of propositions. He meticulously avoids any use of the Parallel Postulate for as long as possible and builds new tools as allowed. It is only at the very end of Book I, after carefully working his way through triangle congruence, parallelism, area and quadrature, that we discover how these pieces fit together to achieve his ultimate goal, the proof of the Pythagorean Theorem and its converse.

Geometric algebra is the topic of Book II in Chapter 8, where readers should experience a newfound appreciation for algebra as it greatly simplifies the propositions of this book. This chapter is particularly helpful for secondary education students since it includes the Law of Cosines. Euclid’s book ends with the quadrature of any polygon, but the chapter ends with a return to Spherical geometry as we consider which figures can be “squared” on a sphere. Chapter 9 briefly covers the topic of similarity (found in Book VI) and ends with a generalization of the Pythagorean Theorem. Chapter 10 explores Euclid’s third book where the focus is squarely on the circle, and our oft-neglected main character gets some long overdue “me time.” Chapter 11 covers Book IV which concerns concurrency points of a triangle, constructions of regular polygons, and results about inscribed and circumscribed circles.

In Chapter 12 we shift to Hyperbolic geometry where we continue to re-examine the concepts of straightness, parallelism, distance and area. Understanding the usefulness of models, we present three models for Hyperbolic geometry, ultimately focusing
Designing a course using this text

On the Poincaré Half-Plane model. In Chapter 13, we give an axiomatic development of Hyperbolic geometry and prove the following surprising results: parallel lines are not everywhere equidistant, the angle sum of a triangle is less than two right angles, rectangles and squares do not exist, AAA is a congruence scheme, and area is a function of angle sums rather than lengths.

We follow this in Chapter 14 with an axiomatic development of finite affine and projective planes, including a lengthy discussion of the history of the development of projective geometry. Chapter 15 covers isometries of the Euclidean and hyperbolic planes. We finish the book with a return to Euclidean geometry to tell the story of four classic construction problems of Greek antiquity and how the nineteenth-century solutions to these problems unambiguously marked the limitations of the Euclidean tools while simultaneously opening new paths ripe for mathematical exploration.

Designing a course using this text

With a healthy amount of both Euclidean and non-Euclidean geometry, instructors are free to choose their focus based on the needs, abilities and mathematical maturity of their students. We outline an option for a course with a majority emphasis on Euclidean topics below, but this can easily be revised to form a minority Euclidean course. Regarding the Euclidean content of the book, rather than simply picking and choosing highlights of the first six books of Euclid’s *Elements*, we have included all of the propositions from the first two books, over half of the propositions from the third book, and nearly all of the fourth book. We do not, however, intend for the instructor to show all of these propositions in class. Our courses are designed to be more interactive than a standard lecture format course, as we find that it helps our students take ownership of the material. We recommend a plan where students rewrite Euclid’s propositions using modern notation and present their updated version to the class. Through this process, students gain skill in both writing and presenting proofs since they must carefully read the proofs in order to separate assumptions from conclusions, and to determine the key definitions, postulates and previous propositions needed in each argument. All of our students, not simply those seeking a degree in secondary education, benefit from these oral presentations (as recommended by Mathematical Association of America [MAA] guidelines). For the construction propositions, we suggest that students reproduce these results using dynamic geometry software. This methodical approach to Euclidean geometry highlights the strength of an axiomatic development and has the added benefit of clearly distinguishing Neutral from non-Neutral propositions. Certainly, some of these propositions can be skipped in the classroom altogether, and they are available in the book as a resource.

While there is more content than can be covered in a semester, this book is, nevertheless, designed to be used for a one-semester course in geometry. Its intended audience is junior/senior undergraduate mathematics majors (be they seeking certification in secondary education or not). One *could* spend the entire semester on the first eleven chapters, but by carefully choosing topics, we expect that an instructor can cover these chapters in roughly eight or nine weeks. To get a sense of how to pace a course, we have included a breakdown of time that we typically spend on each of these chapters. We have taught the course three days a week with 50-minute periods, and two days a week with 75-minute periods.
We fully expect students to read this book. In that spirit, we assign the reading ofChapter 1 before the first class, and we discuss Chapter 1, as well as Chapter 2, on thefirst day. We then cover the first three propositions of Book I and introduce dynamicgeometry software such as Geometer’s Sketchpad® or GeoGebra in class. We have foundthat a basic familiarity with geometric terms facilitates a smooth transition fromunderstanding Heath’s translation of Euclid’s propositions and proofs to updating the proofswith revised notation. Consequently, for roughly a week we have the students presenttheir updated versions of the propositions from Chapter 3, skipping the proofs of theconstruction propositions which are included in a lab assignment utilizing the software.

For Chapter 4, we cover the basic ideas of Spherical geometry and then have thestudents explore which propositions of Neutral geometry still hold in Spherical geometry.Since this chapter lends itself to hands-on exploration, we have our students workin groups to explore the nature of lines, circles and triangles in this strange new universeby utilizing strings, markers, tape and inexpensive plastic balls. The sections onSpherical trigonometry and uniquely Spherical constructions are optional. Likewise,for Chapter 5, we discuss only the basic ideas of Taxicab geometry before ceding the re-mainder of the class to student exploration to determine which propositions of Neutralgeometry still hold in Taxicab geometry. We do not cover all of the proofs in Chap-ter 6 in the classroom, instead choosing to highlight desirable properties of axiomaticsystems as well as Hilbert’s axioms. We spend over a week in Chapter 7, once againhaving students re-write and present select propositions up to I.46. We take the helmto present the Pythagorean Theorem and its converse. Chapters 8 and 9 take about aweek, with “Quadrature on the Sphere” and “A Generalized Pythagorean Theorem”as optional sections. Students present select propositions from Chapter 10 for roughlya week, and we use dynamic software to discuss much of Chapter 11.

- Chapters 1–3 (roughly two weeks)
- Chapters 4–6 (roughly two weeks)
- Chapters 7–9 (roughly two weeks)
- Chapters 10–11 (roughly two weeks)

After Chapter 11, there is considerable flexibility for an instructor to choose a sub-set of the remaining five chapters according to his or her own interests. One of theauthors typically spends the rest of the semester on Chapter 16 (one week), followedby Chapters 12 and 13 (roughly four weeks—skipping some of the proofs from Chap-ter 13), and finishing with Chapter 15 (roughly two weeks—omitting inversions andHyperbolic isometries). The other author replaces Chapter 15 with Chapter 14. It isalso possible to reserve more weeks for the last five chapters of the book by choosingonly the highlights of Chapters 8, 9, 10 and 11.

Common Core State Standards for Mathematics [CCSS]. The followingrecommendations are taken from the Geometry Course report of the Geometry StudyGroup [GSG]. The group was charged by the MAA’s Committee on the UndergraduateProgram in Mathematics [CUPM] with making recommendations about geometry inthe undergraduate mathematics curriculum. Their report is part of the 2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences. Below, we briefly describehow this book addresses each recommended topic.
GSG writes: “To be prepared to teach a geometry course based on CCSS, future teachers should take a college geometry course in which definitions and proof are emphasized. In addition, the course they take should include coverage of the following topics:”

- **Proof**
  We emphasize reading, writing and presenting proofs throughout the book.

- **Transformations**
  In Chapter 15, we prove that any isometry in Neutral geometry can be written as the composition of three or fewer reflections. We then study reflections, rotations, translations and glide reflections in the Euclidean plane. We also consider Euclidean inversions and their role as reflections in the Poincaré Half-plane model of the hyperbolic plane.

- **Parallel Postulate**
  By separating Book I into Chapters 3 and 7, our book is clear on which results in Euclidean geometry depend on the Parallel Postulate. By presenting Spherical, Hyperbolic and projective geometries, we provide multiple two-dimensional geometries in which the Parallel Postulate does not hold.

- **Pythagorean Theorem**
  In addition to Euclid’s proof, we include seven other well-known proofs of this famous theorem in our exercises, including proofs by Bhāskara, Leonardo da Vinci, U.S. President James A. Garfield and Thābit ibn Qurra. We also consider a generalized version of this theorem.

- **Dynamic geometry software**
  We encourage the use of dynamic geometric software throughout the textbook as it provides valuable insight into geometric relationships. We routinely use Geometer’s Sketchpad® or GeoGebra in our courses as well as Spherical Easel and Non-Euclid.

- **Historical perspectives**
  We incorporate historical context throughout the book, particularly regarding the resolution of the controversy surrounding Euclid’s fifth postulate, the development of Hyperbolic, affine and projective geometries, the classic impossible constructions of Greek antiquity, and the development of the mathematics required to prove the impossibility of these constructions.

- **Real-life applications**
  Chapters 11 and 14 include connections between art, architecture and geometry.

Based on these recommendations, an appropriate course could include the following:

- Chapters 1–3
- Chapter 4, sections 1–5
- Chapters 5–7
- Chapter 8, sections 1–2 (while algebraic in nature, helpful for future teachers)
- Chapter 9, sections 1–2
- Chapter 10
- Chapter 11 (optional)
- Chapters 12–13
• Chapter 14 (optional)
• Chapter 15, sections 1–3
• Chapter 16 (optional)

Figure 0.1. Section dependency chart
Good stories have conflict and resolution. The story of geometry is no exception. The characters in this story are geometric objects known since childhood: lines, circles, triangles and squares, to name a few. What conflict could these characters possibly generate? Are we not confident in our deeply ingrained understanding of these fundamental figures? Perhaps you recall a few core facts about triangles—say, the Pythagorean Theorem for right triangles or the 180-degree angle sum of any triangle. While we will revisit these and other well-known results, we will also visit geometric lands where these bedrocks no longer hold, where in some worlds lines are circles, and in others, circles are squares. How is this possible? To paraphrase Walt Whitman, geometry is large; it contains multitudes.

Mathematics, by its very nature, is logical and systematic and, yet, it can still produce results that astonish. One famous case involves Georg Cantor (1845–1918) who, while exploring the nature of infinite sets, documented his incredulity upon discovering that intuition had led him astray, writing to a friend, “Je le vois, mais je ne le crois pas!” (“I see it, but I don’t believe it!”) Cantor was surprised by the conflict that arose when his findings contradicted his expectation that the infinite would play by the same rules as the finite, and yet, he was delighted by the resolution his mathematical reasoning provided. In the same spirit, we aim to present you with a few surprises in geometry that run counter to your intuition.

As Cantor’s story illustrates, surprise requires expectation, and expectation comes from experience. Thus, we need to build our experience and examine our pre-existing assumptions. We start with this fundamental question: How many geometries are there? If you think there is only one then you are in fine company. For 2000 years, there was only one geometry to study, and an ancient Greek mathematician named Euclid was its primary expositor. His book, the Elements, is the most famous and most published mathematics book of all time. Translated into many languages, it was standard reading for students through the centuries and, fittingly, we have chosen it as the starting point for our explorations. It was only relatively recently in the history of mathematics that Euclid’s geometry was found to be just one of many interesting and equally valid geometric worlds. What triggered this revolution? While our two-dimensional figures appear unambiguous, their properties and very nature are more elusive than suggested by first glance. As we will see, a quest to resolve basic questions about the nature of parallel lines was responsible for this seismic paradigm shift.

As we embark on this trip, we first take a closer look at Euclid’s geometry, its origins and its axioms. The trip is all-inclusive; though we have prompted you here for your geometric recollections, we will provide all of the definitions, theorems and proofs necessary for the journey (even the Pythagorean Theorem). At the start, the words of
Euclid's propositions and proofs will be familiar, but the style will be unlike others you have encountered. They are verbose, lacking most of the symbolic language and notation in use today. To better understand a Euclidean proposition and its proof, we suggest that you rewrite it in your own words using standard symbols and notation. Mathematics is a language, and the act of translating this language is a good way to learn how to read it and write it. Austrian Stefan Zweig (1881–1942), the author upon whom *The Grand Budapest Hotel* is based, shared this view. As a young writer he spent several years translating the works of French masters as an improvised apprenticeship in the literary arts. In doing so, he learned the structure of a good book without the pressure of creating the characters, plot or narrative. We echo Zweig's advice to young writers to translate a seasoned author’s work into your language, as this is a reliable method of learning Euclid’s geometry and the art of writing proofs. As a translator, you do not have to create the mathematics, but you will come to understand the logical structure necessary to write clear, correct proofs.

Finally, to state the obvious, we wrote this book to be read by you. To that end, we have included a considerable amount of commentary, history and explanation to help guide you through the story of geometry. Even with the additional narrative, reading a mathematics book is neither an easy nor a passive endeavor. To understand the mathematics you will need to read and then reread the axioms, definitions, theorems and proofs. We find it best to be an active reader with pencil and paper at the ready. Most importantly, as we journey to other strange worlds, keep an open and agile mind and be prepared to abandon preconceived notions as we reconsider and revise our assumptions about geometry and its most familiar objects.
1

The Line and the Circle

1.1 Introduction

Every great story has great characters. *Charlotte’s Web* has Wilbur and Charlotte, *Pride and Prejudice* has Elizabeth Bennet and Mr. Darcy, and *The Hound of the Baskervilles* has Sherlock Holmes and Dr. Watson. The story of geometry is no exception. We start this book by introducing our two main characters: *the line* and *the circle*. Under their
undeniable milquetoast veneer, we will reveal them to be interesting and complex characters as they take our narrative in unexpected directions. We certainly hold no claim to primacy here. In his 1884 novel Flatland: A Romance of Many Dimensions, author Edwin Abbott imbues these objects with personalities and voices (the straight lines are women, the circles are priests). In his 1963 book The Dot and the Line: A Romance in Lower Mathematics, Norton Juster tells the tale of a straight line who falls in love with a dot and then attempts to woo her away from a slothful squiggle. It begins:

Once upon a time there was a sensible straight line who was hopelessly in love with a dot. ‘You’re the beginning and the end, the hub, the core and the quintessence,’ he told her tenderly, but the frivolous dot wasn’t a bit interested, for she only had eyes for a wild and unkempt squiggle who never seemed to have anything on his mind at all. [76]

Both of these stories have been turned into films. The ten-minute animated short, The Dot and the Line, directed by Chuck Jones and Maurice Noble, won the 1965 Oscar for Best Animated Short Film. Both the thirty-four minute Flatland: The Movie (2007) and the ninety-eight minute Flatland: The Film (2007) are based on Abbott’s book. While our main characters are not fictional and will not be given voices, we will learn about them in the same way we come to know all great literary and film characters, largely by observing how they behave.

1.2 Which came first?

The actions of any fictional character are viewed through the cultural and historical lens of the reader. Though our main protagonists are geometric figures, different cultures have historically identified one of these shapes as more fundamental or basic than the other. In particular, certain cultures observe the world and see lines, others find circles. In her book, Ethnomathematics: A Multicultural View of Mathematical Ideas, Marcia Ascher provides us with two such examples [6]. Espousing a rectilinear view of the world, the first is an excerpt from The Stretched String, an essay from the book The Mathematical Experience written by Philip J. Davis and Reuben Hersh in 1981.

“The Stretched String” from The Mathematical Experience

In some primitive cultures there are no number words except one, two, and many. But in every human culture that we will ever discover, it is important to go from one place to another, to fetch water or dig roots. Thus human beings were forced to discover — not once, but over and over again, in each new human life — the concept of the straight line, the shortest path from here to there, the activity of going directly towards something.

In raw nature, untouched by human activity, one sees straight lines in primitive form. The blades of grass or stalks of corn stand erect, the rock falls down straight, objects along a common line of sight are located rectilinearly. But nearly all the straight lines we see around us are human artifacts put there by human labor. The ceiling meets the wall in a straight line, the doors and windowpanes and
tabletops are all bounded by straight lines. Out the window one sees rooftops whose gables and corners meet in straight lines, whose shingles are layered in rows and rows, all straight.

The world, so it would seem, has compelled us to create the straight line so as to optimize our activity, not only by the problem of getting from here to there as quickly and easily as possible, but by other problems as well. For example, when one goes to build a house of adobe blocks, one finds quickly enough that if they are to fit together nicely, their sides must be straight. Thus the idea of a straight line is intuitively rooted in the kinesthetic and the visual imaginations. We feel in our muscles what it is to go straight toward our goal, we can see with our eyes whether someone else is going straight. The interplay of these two sense intuitions gives the notion of straight line a solidity that enables us to handle it mentally as if it were a real physical object that we handle by hand.

By the time a child has grown up to become a philosopher, the concept of a straight line has become so intrinsic and fundamental a part of his thinking that he may imagine it as an Eternal Form, part of the Heavenly Host of Ideals which he recalls from before birth. Or, if his name be not Plato but Aristotle, he imagines that the straight line is an aspect of Nature, an abstraction of a common quality he has observed in the world of physical objects. [30]

Providing an alternative philosophy from a Native American viewpoint, our second excerpt is from the book *Black Elk Speaks*, the story of a holy man of the Sioux tribe as told through John G. Neihardt.

**Excerpt from *Black Elk Speaks***

I came to live here where I am now between Wounded Knee Creek and Grass Creek. Others came too, and we made these little gray houses of logs that you see, and they are square. It is a bad way to live, for there can be no power in a square.

You have noticed that everything an Indian does is in a circle, and that is because the Power of the World always works in circles, and everything tries to be round. In the old days when we were a strong and happy people, all our power came to us from the sacred hoop of the nation, and so long as the hoop was unbroken, the people flourished. The flowering tree was the living center of the hoop, and the circle of the four quarters nourished it. The east gave peace and light, the south gave warmth, the west gave rain, and the north with its cold and mighty wind gave strength and endurance. This knowledge came to us from the outer world with our religion. Everything the Power of the World does is done in a circle. The sky is round, and I have heard that the earth is round like a ball, and so are all the stars. The wind, in its greatest power, whirls. Birds make their nests in circles, for theirs is the same religion as ours. The sun comes forth and goes down again in a circle. The moon does the same, and both are
round. Even the seasons form a great circle in their changing, and always come back again to where they were. The life of a man is a circle from childhood to childhood, and so it is in everything where power moves. Our tepees were round like the nests of birds, and these were always set in a circle, the nation’s hoop, a nest of many nests, where the Great Spirit meant for us to hatch our children.

But the Wasichus (whitemen) have put us in these square boxes. Our power is gone and we are dying, for the power is not in us any more. You can look at our boys and see how it is with us. When we were living by the power of the circle in the way we should, boys were men at twelve or thirteen years of age. But now it takes them very much longer to mature.

Well, it is as it is. We are prisoners of war while we are waiting here. But there is another world. [90]

As we reflect on Davis and Hersh’s claim that the sides of a building must be straight, we may wish to think about the National Museum of the American Indian in Washington, DC. Douglas Joseph Cardinal, the architect of the building, describes his creation as a “a majestic curvilinear form that represents the nurturing female forms of Mother Earth” [36]. Though it is surrounded by buildings that reflect a rectilinear model of physical structure, this building has no straight sides. As an exercise, the reader will compare and contrast the opposing viewpoints represented in these readings.

![National Museum of the American Indian, Washington, D.C.](image)

**Figure 1.2.** National Museum of the American Indian, Washington, D.C.

### 1.3 What is a straight line, anyways?

At the beginning of his book *Experiencing Geometry*, David Henderson challenges his readers to consider the meaning of the word “straight.” In a geometry book whose
1.3 What is a straight line, anyways?

Central characters are the line and the circle, reflecting upon the answer to this question is an excellent way to begin our journey. The following thought exercise is adapted from his book.

**When do you call a line straight?**

The central focus of this thought exercise is to guide the reader in cobbling together a reasonable definition for a straight line, or, at a minimum, to appreciate the difficulty of defining this familiar concept. As a first attempt at an answer, you may try to fit your definition to personal experience. It may help to consider some related practical concerns that arise when constructing lines. For example, suppose you had to mark the first and third base lines on a baseball field, or mark the lines on a volleyball or tennis court. Can you determine a method to produce these straight lines? Once completed, is there a way to check if the lines are straight? Alternatively, suppose you need to construct some object out of 2 × 4 lumber. Assuming you would like to purchase non-warped lumber, how can you determine if a 2 × 4 is straight? More generally, how can you determine if any physical object is straight?

As is true with most concepts, it is often helpful to consider what distinguishes a straight line from its opposite, a non-straight line. If you find yourself relying on the use of a ruler to make this distinction then you must ask yourself how you determined the straightness of the ruler. You may also find yourself relying on the instinctual animal calculation at the heart of the idiom “as the crow flies”: A straight line is the shortest path between two points. Technically, we can only employ this idea if we were to measure all possible paths between two points and then take the shortest. It appears to be a hopeless task and reminiscent of the claims touted by many a merchant. There are only finitely many food markets in any city, or the world for that matter, but claiming to be the world’s best would surely raise some eyebrows. The incredulity factor increases when defining a straight line by the “crow’s method” since there are infinitely many paths between any two points. How can anyone claim to find the shortest path out of infinitely many? Another related question to ponder: If the shortest path between two points is a straight line, then is a straight line between two points always the shortest path?

Oftentimes in this book we will find ourselves considering the symmetries of geometric objects. Saving all formal explanation of symmetry for a later chapter, can you informally describe any symmetries related to a line, and does this help us to define “straight?” One immediate benefit of introducing symmetry is the realization that we can use it to produce a straight line with ease. How? Simply fold a piece of paper.

Lastly, since demonstrative mathematics began with the work of Greek mathematicians, we would be remiss if we did not consider the view of Greek scholars on this matter. To be clear, most high school geometry courses consist entirely of theorems originating in Greece over two millennia ago. Furthermore, the word *mathematics* has its roots in the Greek language. The Greek philosopher Plato (427–347 BCE) believed that a person could not be considered educated without learning mathematics, specifically, the systematic deductions of geometry. There is a well-known story
that at the entryway to his famous Academy stood a sign reading, “Let no one ignorant of geometry enter here.” For Plato, geometric understanding implied an understanding of logic and, hence, the ability to study philosophy. In *Parmenides* he writes: “the round is that of which all the extreme points are equidistant from the centre,” and “the straight is that of which the centre intercepts the view of the extremes.” The modern reader may be satisfied with his description of *round*, but may find that of *straight* lacking. The Greek mathematician Euclid of Alexandria (ca. 325–ca. 265 BCE) wrote the *Elements*, a geometry text that we begin to explore in the following chapter. In this book he writes: “A straight line is a line which lies evenly with the points on itself.” Do these descriptions help us in our quest to determine the meaning of “straight” or to give a definition of “straight line?” Our readers are asked to provide their own description in the exercises.

**Exercises 1.3**

1. Write a short essay to answer the question: “What is a straight line?”

2. Write a short essay (1–2 pages) comparing and contrasting the two readings: “The Stretched String” and the excerpt from *Black Elk Speaks*.

3. Watch *Flatland: The Movie*. Carefully explain how the three-dimensional sphere appears to two-dimensional objects, and hence, how a four-dimensional being would appear to us. What strange abilities would four-dimensional beings have?
Figure 3.1. Proposition I.1 by Oliver Byrne [19]

If we imagine the *Elements* as a play where the first book is Act I, then lines, and the angles, triangles and quadrilaterals they create, play the starring role. Circles are akin to the set crew in this production. They toil in the background and do much of the heavy lifting in Euclid’s proofs, but they are nowhere to be seen on stage in the proposition statements. In general, the statements of Book I are of two types. There
are Euclidean constructions, that is, statements that assert the constructibility of a certain geometric object, for example, an angle bisector or a square. Alternatively, there are statements which explain a relationship between geometric objects, for example, the exterior angle in any triangle must be greater than either opposite interior angle. Often, but not always, the proof of a construction ends with \textit{Q.E.F.}, which is an abbreviation of the Latin phrase \textit{quod erat faciendum}, meaning “that which was to have been done” or “precisely what was required to be done” [39]. The others end with \textit{Q.E.D.}, which stands for \textit{quod erat demonstrandum}, meaning “that which was to have been demonstrated” or “precisely what was required to be proved.” [39] We will use a small square to indicate the end of a proof.

In addition to learning geometric terms and theorems, one of the goals of this chapter is to learn the axiomatic method from its first expositor. Euclid’s book has survived the millennia for good reason: his structure, logic and presentation is hard to beat for this foundational material. He is, quite simply, a master. As we walk through the propositions in his first book, we begin our journey to learn the fundamentals of Euclidean geometry with a focus on developing proof-reading and proof-writing skills.

### 3.1 Propositions I.1 through I.8

The first three propositions are constructions. We give two different proofs for each of the first two propositions. The first proof is Heath’s translation of Euclid’s proof, and the second version follows Euclid’s steps but has updated language and notation. Throughout this chapter, you will be asked to read and update Euclid’s proofs with your own language and notation. As noted in the previous chapter, propositions and proofs reprinted from Heath’s translation of Euclid’s \textit{Elements} are done so with the permission of Dover Publications.

**Proposition I.1.** \textit{On a given finite straight line to construct an equilateral triangle.}

![Figure 3.2. Proposition I.1](image)

**Proof.** Let \( AB \) be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line \( AB \).

With centre \( A \) and distance \( AB \) let the circle \( BCD \) be described; \([\text{Post. 3}]\) again, with centre \( B \) and distance \( BA \) let the circle \( ACE \) be described; \([\text{Post. 3}]\) and from the point \( C \), in which the circles cut one another, to the points \( A, B \) let the straight lines \( CA, CB \) be joined. \([\text{Post. 1}]\)

Now, since the point \( A \) is the centre of the circle \( CDB \), \( AC \) is equal to \( AB \). \([\text{Def. 15}]\)

Again, since the point \( B \) is the centre of the circle \( CAE \), \( BC \) is equal to \( BA \). \([\text{Def. 15}]\)
But \( CA \) was also proved equal to \( AB \); therefore each of the straight lines \( CA, CB \) is equal to \( AB \).

And things which are equal to the same thing are also equal to one another; [C.N. 1] therefore \( CA \) is also equal to \( CB \).

Therefore the three straight lines \( CA, AB, BC \) are equal to one another.

Therefore the triangle \( ABC \) is equilateral; and it has been constructed on the given finite straight line \( AB \).

Being what it was required to do.

Before we give an updated proof, it is important to understand that, in all of his proofs, Euclid starts by stating the givens and then proceeds to explicitly state what he must prove. He then shows what is required, and finally, ends by restating what he has accomplished. It may take a few propositions to grow accustomed to this style. Secondly, Euclid is careful to explain which postulates, common notions and definitions justify each of his steps. Notice, however, that Euclid does not explain why the two circles he constructs will necessarily intersect. His accompanying diagram certainly suggests that such an intersection is inevitable, but Euclid did not intend for his proofs to rely on such drawings. The sketches are only meant as a useful tool to help the reader visualize the proof. Eventually we will have to address this particular oversight by Euclid, and as we proceed, be alert for other logical gaps where an assumption is made without full justification.

In his commentary on the Elements, Proclus details the critiques, and sometimes disparages the critics, of these proofs. Though it seems unimaginable, the relatively straightforward proof of Proposition I.1 generated a full 18 pages of commentary! We will note some of Euclid’s unstated assumptions as we proceed, but we will not be taking a Proclus-sized microscope to these proofs. Instead, in Chapter 4 we highlight the quagmire resulting from unstated assumptions by exploring a different geometry, and in Chapter 6 we detail the eventual modifications made to Euclid’s axioms in the late nineteenth century.

We will follow each Euclidean proof with a box containing the statement of the proposition represented pictorially in miniature, as shown above. These shorthand versions of the propositions are presented in the style of Oliver Byrne (1810-1880), author of the most spectacularly colorful reinvention of Euclid’s work, The First Six Books of The Elements of Euclid in which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners. As demonstrated by the proof of I.1 shown in Figure 3.1, this 1847 book is a translation of Euclid into a hybrid language where shapes of vibrant yellow, red, blue and black substitute for geometric objects such as lines, circles, triangles and rectangles, and symbols such as +, \( \perp \), \( \therefore \) and = take the place of words. (This minimalist aesthetic clearly did not extend to the titling of his work as a simple word count proves the title to be more verbose than his proof of the first proposition.)
Byrne intended to revolutionize the teaching of the standard curriculum of his day with this visual approach to geometry, making it easier for every student to learn Euclid. In his preface, he claims that with his “enticing mode of communicating knowledge, that the Elements of Euclid can be acquired in less than one third the time usually employed, and the retention by the memory is much more permanent.” His use of a primary color palette was entirely in support this goal, for at a glance, one can easily see that two segments or two angles are equal in measure if they share the same color. Sadly, none of his mathematics books were commercially or critically successful during his lifetime, and in particular, this work, now his most famous, was described as a mere “curiosity” by mathematics historian Florian Cajori in 1928 [20].

Though it did not produce a pedagogical revolution, Oliver’s Byrne’s edition of Euclid was lauded in the latter half of the twentieth century for its beauty and artistry in typographic design and printing. That the praise came from outside the mathematical community is not too surprising, for at a distance, the colorful arrangement of woodblock print geometric shapes on many a page gives one the impression of gazing at a work of abstract art in the MoMA or the Pompidou. In particular, Byrne’s style bears an unmistakable likeness to the paintings of Piet Mondrian (1872-1944) and the designs of Frank Lloyd Wright (1867-1959), both artist and architect working over a half century after the appearance of Byrne’s book. Of recent note, the publication of a facsimile reproduction in 2010 has once again revived interest in both the book and the Irish born-and-educated Byrne [19]. For a well-researched account of Byrne’s life we recommend the 2015 article by Hawes and Kolpas, and for a recent revival of Byrne’s style applied to the work of Omar Khayyam (1048–1131), see Kent and Muraki’s 2016 article [68][79].

The colorful miniature propositions given here, our homage to Byrne, utilize a similar set of shapes, colors and algebraic symbols. While the miniature for I.1 requires just one color, propositions that relate two geometric objects, for example I.4, require a larger color palette. We provide miniatures for all Book I propositions and a subset of those from the other books. For ease of reference, we include all of these miniatures in Appendix C. While we do not make any Byrne-like claims as to their time-saving benefits, you may find yourself, like the authors, relying on this Appendix as an at-a-glance visual reference to the Euclidean propositions “for the Greater Ease of Learners” who wish to apply earlier results in later proofs.

We are now ready to provide an updated proof of Proposition I.1.

**Proposition I.1.** On a given finite straight line to construct an equilateral triangle.

**Proof 2.** (Use the diagram given in Figure 3.2 since it is unchanged.)

Let $AB$ be the given line segment.
Using Postulate 3, construct two circles, one centered at $A$, the other at $B$, both with radius $AB$.

Let $C$ be one of the two intersections of these two circles.
Using Postulate 1, draw line segments $AC$ and $BC$.
Since $AC$ and $AB$ are radii of the same circle, by Definition 15, they are equal.
By similar reasoning, $AB = BC$. 


3.1 Propositions I.1 through I.8

By Common Notion 1, we have \( AC = AB = BC \), and hence, we have constructed an equilateral triangle \( \triangle ABC \) as desired.

Take note that our main characters both play a role here. While the first proposition is fundamentally a result about lines, its proof cannot be established without circles. The circles in the proof and represented in the diagram come by way of Postulate 3, giving us the ability to construct a circle at a given center with a given radius, and essentially, providing every geometer with a theoretical collapsible compass. As mentioned in the previous chapter, compasses are either collapsible or rigid. Clearly, anything that can be constructed with a collapsible compass can also be constructed with a rigid one. This next proposition proves the converse, that is, anything that can be constructed with a rigid compass can also be constructed with a collapsible compass. Thus, the two tools are mathematically equivalent. You may wonder why Euclid did not choose to make his third postulate stronger, and hence, avoid the need to prove this second proposition. In general, mathematicians agree with Aristotle (384–322 BCE) that “other things being equal, that proof is the better which proceeds from the fewer postulates.” [15] More generally, an economy of assumptions makes for a stronger theory. After this proof and for the remainder of the book (including when we explore other geometries), we will assume that we have a rigid compass.

**Proposition I.2.** To place at a given point [as an extremity] a straight line equal to a given straight line.

![Figure 3.3. Proposition I.2](image-url)

**Proof.** Let \( A \) be the given point, and \( BC \) the given straight line.

Thus it is required to place at the point \( A \) [as an extremity] a straight line equal to the given straight line \( BC \).

From the point \( A \) to the point \( B \) let the straight line \( AB \) be joined; [Post. 1] and on it let the equilateral triangle \( DAB \) be constructed. [I.1]

Let the straight lines \( AE, BF \) be produced in a straight line with \( DA, DB \); [Post. 2] with centre \( B \) and distance \( BC \) let the circle \( CGH \) be described; [Post. 3] and again, with centre \( D \) and distance \( DG \) let the circle \( GKL \) be described. [Post. 3]

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1The square brackets within the statement of a proposition indicate material added by Heath to clarify the Greek text.
Then, since the point \( B \) is the centre of the circle \( CGH \), \( BC \) is equal to \( BG \).
Again, since the point \( D \) is the centre of the circle \( GKL \), \( DL \) is equal to \( DG \).
And in these \( DA \) is equal to \( DB \); therefore the remainder \( AL \) is equal to the remainder \( BG \). [C.N. 3]
But \( BC \) was also proved equal to \( BG \); therefore each of the straight lines \( AL, BC \) is equal to \( BG \).
And things which are equal to the same thing are also equal to one another; [C.N. 1] therefore \( AL \) is also equal to \( BC \).
Therefore at the given point \( A \) the straight line \( AL \) is placed equal to the given straight line \( BC \).
Being what it was required to do.

Proof 2. (Use the diagram given in Figure 3.3 since it is unchanged.)
Let \( A \) be the given point and \( BC \) be the given line segment.
Using Postulate 1, draw segment \( AB \).
With Proposition 1, construct an equilateral triangle using \( AB \) as a side, let the other vertex be labeled \( D \).
Draw rays \( \overrightarrow{DA} \) and \( \overrightarrow{DB} \) with Postulate 2.
With Postulate 3, construct a circle with center \( B \) and radius \( BC \).
Label its intersection with \( \overrightarrow{DB} \) as \( G \).
Construct a second circle with center \( D \) and radius \( DG \).
Label the intersection of this circle with \( \overrightarrow{DA} \) as \( L \).
We claim that \( AL = BC \) as follows:
Since \( DL \) and \( DG \) are both radii of the same circle, they must be equal. [Def. 15]
By construction, \( DA = DB \), thus by Common Notion 3, we have \( AL = BG \).
But \( BG \) and \( BC \) are radii of the same circle so \( BG = BC \). [Def. 15]
Hence by Common Notion 1, we have \( AL = BC \) as desired.

With a rigid compass comes the ability to transfer a length in the plane, which in turn produces the ability to add one segment to another, or subtract a shorter segment from a longer as we see in the next proposition. This is addition and subtraction, but not with numbers, plus signs or minus signs. In fact, the first plus sign in this chapter does not appear until page 38, and it is only the second equation sporting the symbol in the first 38 pages of the book. Arithmetic and algebraic symbols are our mathematical language, but they would be Greek to Euclid, so to speak. For the simple fact that the mathematical notation we use today developed over the millennium and a half after his lifetime, Euclid used words to convey his ideas. We favor algebraic symbols. So, when Euclid writes the lengthy, “Let \( AB, CD \) be the two given unequal straight lines, and let \( AB \) be the greater.”, we substitute “Let \( AB > CD \.” We cannot help but include our mathematical language when we give an updated translation of a Euclidean proof. Keep this in mind, but don’t lose sight of the fact that geometry is the scope through which Euclid views arithmetic and algebra throughout the \textit{Elements}. 
**Proposition I.3.** Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

![Figure 3.4. Proposition I.3](image)

**Proof.** Let \( AB, C \) be the two given unequal straight lines, and let \( AB \) be the greater of them.

Thus it is required to cut off from \( AB \) the greater a straight line equal to \( C \) the less.

At the point \( A \) let \( AD \) be placed equal to the straight line \( C \); [I.2] and with centre \( A \) and distance \( AD \) let the circle \( DEF \) be described. [Post. 3]

Now, since the point \( A \) is the centre of the circle \( DEF \), \( AE \) is equal to \( AD \). [Def. 15] But \( C \) is also equal to \( AD \).

Therefore each of the straight lines \( AE, C \) is equal to \( AD \); so that \( AE \) is also equal to \( C \). [C.N. 1]

Therefore, given the two straight lines \( AB, C \), from \( AB \) the greater \( AE \) has been cut off equal to \( C \) the less.

Being what it was required to do. \( \square \)

### Definition 3.1.

Two triangles \( \triangle ABC \) and \( \triangle DEF \) are congruent, denoted \( \triangle ABC \cong \triangle DEF \), if both their corresponding sides and their corresponding angles are equal, that is, \( AB = DE, BC = EF, AC = DF \) and \( \angle ABC = \angle DEF, \angle BCA = \angle EFD, \angle BAC = \angle EDF \).
It is important to note that the order of the vertices in each triangle must correspond to the congruence. That is, if \( \triangle ABC \cong \triangle GHI \), then we know \( AB = GH \), \( BC = HI \), \( AC = GI \) and \( \angle A = \angle G \), \( \angle B = \angle H \), \( \angle C = \angle I \). For notational ease, we will abbreviate \( \angle ABC \) as \( \angle B \) whenever there is only one angle to be found at \( B \), and hence, no possibility for confusion. For general polygons, two polygons are said to be congruent if their corresponding sides and angles are all equal.

Proposition I.4 is commonly known as the SAS (side-angle-side) congruence scheme since Euclid proves that the equivalence of two sides and their included angles is enough to guarantee the congruence of two triangles. Before the end of this chapter we will encounter three other congruence schemes for triangles, SSS [I.8] (side-side-side), ASA [I.26] (angle-side-angle) and AAS [I.26] (angle-angle-side).

Proposition I.4 [SAS]. If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

![Figure 3.5. Proposition I.4 SAS](image)

**Proof.** Let \( ABC \), \( DEF \) be two triangles having the two sides \( AB \), \( AC \) equal to the two sides \( DE \), \( DF \) respectively, namely \( AB \) to \( DE \) and \( AC \) to \( DF \), and the angle \( BAC \) equal to the angle \( EDF \).

I say that the base \( BC \) is also equal to the base \( EF \), the triangle \( ABC \) will be equal to the triangle \( DEF \), and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle \( ABC \) to the angle \( DEF \), and the angle \( ACB \) to the angle \( DFE \).

For, if the triangle \( ABC \) be applied to the triangle \( DEF \), and if the point \( A \) be placed on the point \( D \) and the straight line \( AB \) on \( DE \), then the point \( B \) will also coincide with \( E \), because \( AB \) is equal to \( DE \).

Again, \( AB \) coinciding with \( DE \), the straight line \( AC \) will also coincide with \( DF \), because the angle \( BAC \) is equal to the angle \( EDF \); hence the point \( C \) will also coincide with the point \( F \), because \( AC \) is again equal to \( DF \).

But \( B \) also coincided with \( E \); hence the base \( BC \) will coincide with the base \( EF \), and will be equal to it. [C.N. 4]

Thus the whole triangle \( ABC \) will coincide with the whole triangle \( DEF \), and will be equal to it. [C.N. 4]

And the remaining angles will also coincide with the remaining angles and will be equal to them, the angle \( ABC \) to the angle \( DEF \), and the angle \( ACB \) to the angle \( DFE \). [C.N. 4]

Therefore etc. Q.E.D.
There are a few items of interest in the proof of Proposition I.4. First, it is within the body of the proof that Euclid claims that there is exactly one line segment joining any two points. Therefore, he assumes that a line between two points, as given by Postulate 1, is unique. This is another Euclidean omission to be addressed in a later chapter. More importantly, Euclid employs a proof technique that only appears twice in Book I, and it raises some questions. Essentially, Euclid picks up $\triangle ABC$ and “applies,” or superposes, it on top of $\triangle DEF$. He then argues that the triangles will precisely match up, and thus, they coincide. This is called a **proof by superposition**, and we will see this technique again in Proposition I.8 [SSS]. One of Euclid’s unstated assumptions here is that moving these triangles in the plane does not deform them like a funhouse mirror would. Although related to Common Notion 4, none of Euclid’s postulates or common notions allows him to “apply” one triangle to another. This is another example of Euclid’s omissions that we will need to address in a subsequent chapter. Lastly, it is in this proposition that Euclid calls one of the segments forming the trilateral its base, and the remaining segments its sides, as a convenient naming scheme.

The most difficult of the early propositions is the fifth, which subsequently came to be known as *pons asinorum*, a Latin phrase translated as *bridge of fools* or *ass’s bridge*. The reason for its nickname is not entirely clear, but it is speculated that the proof was less of a bridge and more of dead-end for poor geometry students. We will include two proofs, the first from the *Elements*.

**Proposition I.5.** *In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.*

**Proof.** Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$; and let the straight lines $BD, CE$ be produced further in a straight line with $AB, AC$. [Post. 2]

I say that the angle $ABC$ is equal to the angle $ACB$, and the angle $CBD$ to the angle $BCE$.

Let a point $F$ be taken at random on $BD$; from $AE$ the greater let $AG$ be cut off equal to $AF$ the less; [I.3] and let the straight lines $FC, GB$ be joined. [Post. 1]
Then, since $AF$ is equal to $AG$ and $AB$ to $AC$, the two sides $FA$, $AC$ are equal to the two sides $GA$, $AB$, respectively; and they contain a common angle, the angle $FAG$.

Therefore the base $FC$ is equal to the base $GB$, and the triangle $AFC$ is equal to the triangle $AGB$, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle $ACF$ to the angle $ABG$, and the angle $AFC$ to the angle $AGB$. [I.4]

And, since the whole $AF$ is equal to the whole $AG$, and in these $AB$ is equal to $AC$, the remainder $BF$ is equal to the remainder $CG$.

But $FC$ was also proved equal to $GB$; therefore the two sides $BF$, $FC$ are equal to the two sides $CG$, $GB$ respectively; and the angle $BFC$ is equal to the angle $CGB$, while the base $BC$ is common to them; therefore the triangle $BFC$ is also equal to the triangle $CGB$, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle $FBC$ is equal to the angle $GCB$, and the angle $BCF$ to the angle $CBG$.

Accordingly, since the whole angle $ABG$ was proved equal to the angle $ACF$, and in these the angle $CBG$ is equal to the angle $BCF$, the remaining angle $ABC$ is equal to the remaining angle $ACB$; and they are at the base of the triangle $ABC$.

But the angle $FBC$ was also proved equal to the angle $GCB$; and they are under the base.

Therefore etc. Q.E.D. □

Notice that Euclid writes that two triangles are “equal” when we would more precisely describe them as “congruent.” This becomes particularly important later in Book I when Euclid uses the term “equal” to mean that the triangles have the same area (but are not necessarily congruent). In general, it is clear from Euclid’s context which meaning is correct. Also, when writing an updated version of the proof the reader may be tempted to use Common Notion 3 in order to claim that angles $\angle FBC$ and $\angle GCB$ are equal. Since we are only allowed to rely on the axioms and any earlier propositions, such a claim is invalid since it rests on a geometric result that has not been shown yet. It is not until Proposition I.13 that Euclid proves that a straight line standing upon a straight line makes either two right angles or angles whose sum equals two right angles.

The second proof offers an alternative approach to the first part of the theorem and is written with updated notation. Proclus attributes this proof to Pappus of Alexandria (ca. 290–ca. 350 BCE).

**Proof 2.** Let $\triangle ABC$ be an isosceles triangle with $AB = AC$.

We will think of this triangle in two different ways, namely as both $\triangle ABC$ and $\triangle ACB$.

Since $AB = AC$, we clearly have $AC = AB$.

We also have $\angle BAC = \angle CAB$. 


Thus by I.4 [SAS], we have \( \triangle ABC \cong \triangle ACB \) and hence \( \angle ABC = \angle ACB \) as desired.

The next proposition is the converse of the previous. A statement of the form “If \( P \), then \( Q \)” is called an implication, and its converse is the statement “If \( Q \), then \( P \).” In general, the validity of an implication does not imply the validity of its converse. For example, consider the statement: If it is raining, then I carry an umbrella. This statement merely implies that I am well-prepared for rain. Its converse is: If I carry an umbrella, then it is raining. The validity of this statement, on the other hand, would imply that I am, in fact, a rain god. (See Exercise 2.3.3 for another example.) When both an implication and its converse are valid, Euclid often follows the proof of the implication with its converse.

The proof of Proposition I.6 is our first formal example of the proof technique known as a proof by contradiction. (We say “formal” since we rather informally used this technique in our discussion of Common Notion 5 in the previous chapter.) When proving the implication “If \( P \), then \( Q \)” with this technique, we begin by assuming \( P \) and the negation of \( Q \), and show that this leads to an absurdity, or contradiction. This technique, also called reductio ad absurdum, offers an alternative to a direct proof of the implication “If \( P \), then \( Q \)” where \( P \) is assumed to be true and \( Q \) must be shown to be true. In the proof of Proposition I.6, Euclid assumes that two angles in a triangle are equal (\( P \)) and the sides subtending the angles are not equal (negation of \( Q \)), and then finds a contradiction resulting from these assumptions. More specifically, he assumes \( \angle ABC = \angle ACB \) and \( AB \neq AC \), and then derives a contradiction.

**Proposition I.6.** If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.
Proof. Let $ABC$ be a triangle having the angle $ABC$ equal to the angle $ACB$; I say that the side $AB$ is also equal to the side $AC$.

For, if $AB$ is unequal to $AC$, one of them is greater.

Let $AB$ be greater; and from $AB$ the greater let $DB$ be cut off equal to $AC$ the less; let $DC$ be joined.

Then, since $DB$ is equal to $AC$, and $BC$ is common, the two sides $DB, BC$ are equal to the two sides $AC, CB$ respectively; and the angle $DBC$ is equal to the angle $ACB$; therefore the base $DC$ is equal to the base $AB$, and the triangle $DCB$ will be equal to the triangle $ACB$, the less to the greater: which is absurd.

Therefore $AB$ is not unequal to $AC$; it is therefore equal to it.

Therefore etc. Q.E.D.

When working with an isosceles triangle and trying to recall the distinction between Propositions I.5 and I.6 in order to justify some step in a proof, it may help to know that Euclid never uses Proposition I.6 in any proof for the remainder of Book I.

The following proposition gives the result Euclid needs to prove the congruence scheme SSS of Proposition I.8. We call Proposition I.7 a lemma as it is not particularly interesting in its own right, but is helpful when proving a more substantial theorem.

As with the previous proposition, Euclid employs a proof by contradiction.

**Proposition I.7.** Given two straight lines constructed on a straight line [from its extremities] and meeting in a point, there cannot be constructed on the same straight line [from its extremities], and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

**Proof.** For, if possible, given two straight lines $AC, CB$ constructed on the straight line $AB$ and meeting at the point $C$, let two other straight lines $AD, DB$ be constructed on the same straight line $AB$, on the same side of it, meeting in another point $D$ and equal to the former two respectively, namely each to that which has the same extremity with it, so that $CA$ is equal to $DA$ which has the same extremity $A$ with it, and $CB$ to $DB$ which has the same extremity $B$ with it; and let $CD$ be joined.
Then, since $AC$ is equal to $AD$, the angle $ACD$ is also equal to the angle $ADC$; [I.5] therefore the angle $ADC$ is greater than the angle $DCB$; therefore the angle $CDB$ is much greater than the angle $DCB$.

Again, since $CB$ is equal to $DB$, the angle $CDB$ is also equal to the angle $DCB$. But it was also proved much greater than it: which is impossible.

Therefore etc. Q.E.D.

Euclid’s proof of I.7 depends upon the configuration of points $C$ and $D$. He assumes that $D$ lies outside triangle $\triangle ABC$, but we must consider the alternative case where $D$ lies inside the triangle, as illustrated in Figure 3.10. We leave it to the reader as Exercise 3.1.5 to find the contradiction in this case, and take this opportunity to warn the reader about the tendency to overlook the subtleties in a proof when relying on diagrams. We must be careful to read these proofs with a critical eye since Euclid does not always provide a proof for every possible case. It was the tradition of the Greek geometers to give one case, usually the most difficult, and to leave any other cases to the reader.

**Proposition I.8 [SSS].** If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.
Proof. Let $ABC$ and $DEF$ be two triangles having the two sides $AB, AC$ equal to the two sides $DE, DF$ respectively, namely $AB$ to $DE$, and $AC$ to $DF$; and let them have the base $BC$ equal to the base $EF$.

I say that the angle $BAC$ is also equal to the angle $EDF$.

For, if the triangle $ABC$ is applied to the triangle $DEF$, and if the point $B$ be placed on the point $E$ and the straight line $BC$ on $EF$, the point $C$ will also coincide with $F$, because $BC$ equals $EF$.

Then, $BC$ coinciding with $EF$, $BA, AC$ will also coincide with $ED, DF$; for, if the base $BC$ coincides with the base $EF$, and the sides $BA, AC$ do not coincide with $ED, DF$ but fall beside them as $EG, GF$, then given two straight lines constructed on a straight line [from its extremities] and meeting in a point, there will have been constructed on the same straight line [from its extremities], and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the extremity with it.

But they cannot be so constructed. [I.7]

Therefore it is not possible that, if the base $BC$ is applied to the base $EF$, the sides $BA, AC$ should not coincide with $ED, DF$; they will therefore coincide, so that the angle $BAC$ will also coincide with the angle $EDF$, and will be equal to it.

If therefore etc. Q.E.D.

A careful reader may notice that Euclid does not specifically state that points $D$ and $G$ are on the same side of the plane determined by line $EF$. Since this requirement is part of the hypotheses of Proposition I.7, Euclid clearly means that the points meet this condition (as shown in his diagram for I.8), else he could not use I.7.

Before we begin the exercises, we introduce the following two definitions.

**Definition 3.2.** The point $M$ of the line segment $AB$ such that $AM = MB$ is called the **midpoint** of $AB$.

**Definition 3.3.** The line segment joining a vertex of a triangle to the midpoint of the opposite side is called a **median**.
Exercises 3.1

1. Following the examples of Proposition I.1 and Proposition I.2, give an updated version of Euclid’s proof of Proposition I.5. Be sure to justify each step, substitute mathematical symbols where appropriate, and include helpful diagrams as needed. Compare Euclid’s proof with that of Pappus. Which do you prefer? Why? Why do you think Euclid chose his own proof instead of the shorter version?

2. Give an updated version of Euclid’s proof of Proposition I.6. Be sure to justify each step, substitute mathematical symbols where appropriate, and include helpful diagrams as needed. When writing a proof by contradiction it is helpful to the reader to declare your intention at the start. This can be accomplished by writing Proof (by contradiction): at the start, or announcing your intended direction with the first sentence of the proof.

3. Give an updated version of Euclid’s proof of Proposition I.7. Be sure to justify each step, substitute mathematical symbols where appropriate, and include helpful diagrams as needed.

4. Euclid’s proof of Proposition I.7 uses the law of trichotomy as well as two other properties of inequality which are not given as common notions. One of them is the transitivity of inequality and another is a closely related property of inequality. [Hint for the latter: If \( a = b \) and \( b < c \) . . .] Use your proof in Exercise 3 to give these missing properties.

5. For the proof of Proposition I.7, consider the alternative configuration where \( D \) lies inside \( \triangle ABC \). We will reproduce Proclus’ contradiction for this case by extending \( AC \) and \( AD \) to points \( E \) and \( F \), respectively, as illustrated in Figure 3.10. Show that angle \( \angle BDC \) is both greater than \( \angle BCD \), and equal to angle \( \angle BCD \), which is clearly not possible.

6. Follow the instructions in Exercise 3 for Proposition I.8.

7. A kite is a convex quadrilateral with two pairs of congruent adjacent sides. The convex quadrilateral \( ABCD \) shown in Figure 3.13 is a kite since \( AB = BC \) and \( AD = CD \). Prove that the diagonals of a kite intersect at right angles.
8. Two quadrilaterals $ABCD$ and $EFGH$ are congruent if both their corresponding sides and their corresponding angles are equal, that is, $AB = EF, BC = FG, CD = GH$ and $AD = EH$, and $\angle A = \angle E, \angle B = \angle F, \angle C = \angle G$ and $\angle D = \angle H$, as illustrated in Figure 3.14.

Though Euclid does not give any congruence schemes for quadrilaterals, we add congruence scheme SASAS for convex quadrilaterals here. Prove congruence scheme SASAS for convex quadrilaterals. In other words, assuming that $AB = EF, BC = FG, CD = GH, \angle B = \angle F$ and $\angle C = \angle G$, prove that $\angle A = \angle E, \angle D = \angle H$ and $AD = EH$.

Note: for the next two exercises, we need the fact that if $x = y$, then $\frac{1}{2}x = \frac{1}{2}y$. In the proof of Proposition I.37, Euclid claims, but does not prove, the following: “But the halves of equal things are equal to one another.” We will wait until Chapter 7 to formally prove this.

9. Prove that the medians to the equal sides of an isosceles triangle are equal to each other.

10. Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle is also isosceles.

11. Consider the isosceles triangle $\triangle ABC$ with $AB = AC$. Choose points $D$ and $E$ on $AB$ and $AC$, respectively, such that $AD = AE$. Draw a corresponding picture and then prove that $CD = BE$.

12. Consider the isosceles triangle $\triangle ABC$ with $AB = AC$. Choose points $D$ and $E$ on side $BC$ such that $BD = CE$. Draw a corresponding picture and then prove that $AD = AE$.

### 3.2 Propositions I.9 through I.15

The next four propositions are all constructions. Proposition I.9 gives us the ability to bisect any given angle while Proposition I.10 gives us the ability to bisect any given
3.2 Propositions I.9 through I.15

segment. In Propositions I.11 and I.12 we construct perpendicular lines to a given line segment, the first from a point on the segment, the second from a point not on it. These four along with Proposition I.23 (which allows us to copy a given angle onto a segment) and Proposition I.31 (in which we construct a line parallel to a given line through a point not on the given line), provide a basic toolbox for Euclid. It may surprise the reader that while bisecting any angle is a straightforward task, trisecting it, or dividing it into three equal angles, is not. It would take until 1837 before Pierre Wantzel (1814–1848) proved that it is, in fact, impossible to trisect every angle using only a straightedge and compass. Finally, while these next four propositions are constructions, Euclid’s reliance on previous propositions within the body of each proof makes the resulting steps inefficient and impractical as a construction algorithm. In the exercises, the reader is asked to provide a more succinct set of instructions for each construction and provide the corresponding proof.

**Proposition I.9.** *To bisect a given rectilineal angle.*

![Figure 3.15. Proposition I.9](image)

**Proof.** Let the angle $BAC$ be the given rectilineal angle.

Thus it is required to bisect it.

Let a point $D$ be taken at random on $AB$; let $AE$ be cut off from $AC$ equal to $AD$; [I.3] let $DE$ be joined, and on $DE$ let the equilateral triangle $DEF$ be constructed; let $AF$ be joined.

I say that the angle $BAC$ has been bisected by the straight line $AF$.

For, since $AD$ is equal to $AE$, and $AF$ is common, the two sides $DA$, $AF$ are equal to the two sides $EA$, $AF$ respectively.

And the base $DF$ is equal to the base $EF$; therefore the angle $DAF$ is equal to the angle $EAF$. [I.8]

Therefore the given rectilineal angle $BAC$ has been bisected by the straight line $AF$.

Q.E.F.

The line $AF$ in Proposition 9 is called the **angle bisector** of angle $\angle BAC$.

**Proposition I.10.** *To bisect a given finite straight line.*
Proof. Let $AB$ be the given finite straight line. Thus it is required to bisect the finite straight line $AB$.

Let the equilateral triangle $ABC$ be constructed on it, [I.1] and let the angle $ACB$ be bisected by the straight line $CD$; [I.9]

I say that the straight line $AB$ has been bisected at the point $D$.

For, since $AC$ is equal to $CB$, and $CD$ is common, the two sides $AC, CD$ are equal to the two sides $BC, CD$ respectively; and the angle $ACD$ is equal to the angle $BCD$; therefore the base $AD$ is equal to the base $BD$. [I.4]

Therefore the given finite straight line $AB$ has been bisected at $D$.

Q.E.F.

\[ \begin{array}{c}
\text{I.10} \\
\text{Given } A \leftarrow a \rightarrow B \text{ construct } A \leftarrow \frac{a}{2} M \rightarrow \frac{a}{2} B
\end{array} \]

While it is a subtle point, Euclid does not justify why the angle bisector for $\angle ACB$ intersects side $AB$ at a point $D$. We will add this to our list of unstated assumptions that will eventually need to be addressed. The line $CD$ in the proof is called the perpendicular bisector of $AB$, which we define in general as follows. Additionally, using updated notation, we write $CD \perp AB$ to indicate that $CD$ is perpendicular to $AB$.

**Definition 3.4.** Given a line segment $AB$, the straight line through its midpoint that is also perpendicular to $AB$ is called its perpendicular bisector.

The following theorem is not found in the *Elements* and is our first example of a **biconditional statement**, namely, a statement of form “$P$ if and only if $Q$.” which is abbreviated as “$P$ iff $Q$.” This type of statement includes both the implication “If $P$, then $Q$.” and its converse “If $Q$, then $P$.” The proof of a biconditional statement typically consists of separate proofs of each implication. Alternatively, a proof which utilizes a sequence of other logically equivalent biconditional statements can be given. This is often the case with proofs from high-school algebra. We leave the proof of this theorem as two exercises, one for each implication. Also, two extra notes about distance are in order. While we will discuss the notion of distance more formally in later chapters, for now please take the distance between two points as the length of the line segment between them. Lastly, **equidistant** means equally distant, or of the same distance.

**Theorem 3.5.** Given a line segment $AB$, a point $C$ is equidistant from $A$ and $B$ (that is, $CA = CB$), if and only if $C$ lies on the perpendicular bisector of $AB$. 
Proposition I.11. To draw a straight line at right angles to a given straight line from a given point on it.

Proof. Let $AB$ be the given straight line, and $C$ the given point on it.

Thus it is required to draw from the point $C$ a straight line at right angles to the straight line $AB$.

Let a point $D$ be taken at random on $AC$; let $CE$ be made equal to $CD$; [I.3] on $DE$ let the equilateral triangle $FDE$ be constructed, [I.1] and let $FC$ be joined;

I say that the straight line $FC$ has been drawn at right angles to the given straight line $AB$ from $C$ the given point on it.

For, since $DC$ is equal to $CE$, and $CF$ is common, the two sides $DC, CF$ are equal to the two sides $EC, CF$ respectively; and the base $DF$ is equal to the base $FE$; therefore the angle $DCF$ is equal to the angle $ECF$; [I.8] and they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [Def. 10] therefore each of the angles $DCF, FCE$ is right.

Therefore the straight line $CF$ has been drawn at right angles to the given straight line $AB$ from the given point $C$ on it.

Q.E.F.

Proposition I.12. To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Proof. Let $AB$ be the given infinite straight line, and $C$ the given point which is not on it; thus it is required to draw to the given infinite straight line $AB$, from the given point $C$ which is not on it, a perpendicular straight line.

For let a point $D$ be taken at random on the other side of the straight line $AB$, and with centre $C$ and distance $CD$ let the circle $EFG$ be described; [Post. 3] let the straight line $EG$ be bisected at $H$, [I.10] and let the straight lines $CG, CH, CE$ be joined. [Post. 1]

I say that $CH$ has been drawn perpendicular to the given infinite straight line $AB$ from the given point $C$ which is not on it.
For, since $GH$ is equal to $HE$, and $HC$ is common, the two sides $GH$, $HC$ are equal to the two sides $EH$, $HC$ respectively; and the base $CG$ is equal to the base $CE$; therefore the angle $CHG$ is equal to the angle $EHC$. [I.8]

And they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands. [Def. 10]

Therefore $CH$ has been drawn perpendicular to the given infinite straight line $AB$ from the given point $C$ which is not on it.

Q.E.F.

When constructing a perpendicular line and trying to recall the distinction between Propositions I.11 and I.12 in order to justify some step in a proof, it may help to know that Euclid never uses Proposition I.12 in any proof for the remainder of Book I.

The next few propositions concern the angles created by intersecting lines. Here are a few relevant definitions giving updated terminology.

**Definition 3.6.** If $A$ is a point not on the straight line $CD$ and $B$ is a point on $CD$ between $C$ and $D$, then $\angle ABD$ and $\angle ABC$ are said to be **supplementary angles**.

**Definition 3.7.** If $AB$ and $CD$ are straight lines intersecting at point $E$, then angles $\angle AEC$ and $\angle BED$ are said to be **vertical angles**.
3.2 Propositions I.9 through I.15

Proposition I.13 states that supplementary angles sum to two right angles and Proposition I.14 is its converse. While these could be combined to form an iff statement, Euclid rarely utilizes biconditional statements in the Elements, the first appearing in Book III. Though these two propositions may seem fairly obvious, the more interesting, I.15, is often referred to as the “Vertical Angle Theorem” as it asserts the equivalence of vertical angles.

**Proposition I.13.** If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{figure3_21.png}
  \caption{Proposition I.13}
\end{figure}

**Proof.** For let any straight line $AB$ set up on the straight line $CD$ make the angles $CBA$, $ABD$;

I say that the angles $CBA$, $ABD$ are either two right angles or equal to two right angles.

Now, if the angle $CBA$ is equal to the angle $ABD$, they are two right angles. [Def. 10]

But, if not, let $BE$ be drawn from the point $B$ at right angles to $CD$; [I.11] therefore the angles $CBE$, $EBD$ are two right angles.

Then, since the angle $CBE$ is equal to the two angles $CBA$, $ABE$, let the angle $EBD$ be added to each; therefore the angles $CBE$, $EBD$ are equal to the three angles $CBA$, $ABE$, $EBD$. [C.N. 2]

Again, since the angle $DBA$ is equal to the two angles $DBE$, $EBA$, let the angle $ABC$ be added to each; therefore the angles $DBA$, $ABC$ are equal to the three angles $DBE$, $EBA$, $ABC$. [C.N. 2]

But the angles $CBE$, $EBD$ were also proved equal to the same three angles; and things which are equal to the same thing are also equal to one another; [C.N. 1] therefore the angles $CBE$, $EBD$ are also equal to the angles $DBA$, $ABC$.

But the angles $CBE$, $EBD$ are two right angles; therefore the angles $DBA$, $ABC$ are also equal to two right angles.

Therefore etc. Q.E.D. 
\[\Box\]
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I.13 \[ \alpha \beta \Rightarrow \alpha + \beta = 2 \]

Notice that in this proof, Euclid assumes that \( \angle CBA \) is acute and hence \( \angle CBE = \angle CBA + \angle ABE \). Certainly this matches his diagram, but from the start he could have chosen \( \angle ABD \) to be acute and drawn the resulting diagram. Upon reflection, it is clear that either choice would have led to the same conclusion and only the letters and diagram would be different. To acknowledge that there is another choice that can be made and note the resulting logical irrelevance of the choice, within the body of the proof we preface the choice with the phrase “Without loss of generality, assume...” which is abbreviated with WLOG. In an updated version of this particular proof we could write, “WLOG, assume \( \angle CBA \) is acute.” One last note for I.13 concerns the new symbol found in its miniature representation shown in the box. Here, the symbol \( \| \) is used to denote a right angle.

**Proposition I.14.** If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

![Figure 3.22. Proposition I.14](image)

**Proof.** For with any straight line \( AB \), and at the point \( B \) on it, let the two straight lines \( BC, BD \) not lying on the same side make the adjacent angles \( ABC, ABD \) equal to two right angles;

I say that \( BD \) is in a straight line with \( CB \).

For, if \( BD \) is not in a straight line with \( BC \), let \( BE \) be in a straight line with \( CB \).

Then, since the straight line \( AB \) stands on the straight line \( CBE \), the angles \( ABC, ABE \) are equal to two right angles; [I.13]

But the angles \( ABC, ABD \) are also equal to two right angles; therefore the angles \( CBA, ABE \) are equal to the angles \( CBA, ABD \). [Post. 4 and C.N. 1]

Let the angle \( CBA \) be subtracted from each; therefore the remaining angle \( ABE \) is equal to the remaining angle \( ABD \), [C.N. 3] the less to the greater: which is impossible. Therefore \( BE \) is not in a straight line with \( CB \).

Similarly we can prove that neither is any other straight line except \( BD \).

Therefore \( CB \) is in a straight line with \( BD \).

Therefore etc. Q.E.D.
It is a bit odd that Euclid bothers to write “Similarly we can prove that neither is any other straight line except $BD$.” For when he assumes that $BD$ is not in a straight line with $CB$, Postulate 2 tells us that we can extend $CB$. Thus, there must exist an $E$ such that $BE$ is in a straight line with $CB$. Since the proof works regardless of whether or not $E$ lies within $\angle ABD$, perhaps this is an acknowledgment by Euclid that his diagram does not represent all possible cases.

**Proposition I.15 [Vertical Angle Theorem].** If two straight lines cut one another, they make the vertical angles equal to one another.

**Figure 3.23.** Proposition I.15

**Proof.** For let the straight lines $AB, CD$ cut one another at the point $E$; I say that the angle $AEC$ is equal to the angle $DEB$, and the angle $CEB$ to the angle $AED$.

For, since the straight line $AE$ stands on the straight line $CD$, making the angles $CEA, AED$, the angles $CEA, AED$ are equal to two right angles. [I.13]

Again, since the straight line $DE$ stands on the straight line $AB$, making the angles $AED, DEB$, the angles $AED, DEB$ are equal to two right angles. [I.13]

But the angles $CEA, AED$ were also proved equal to two right angles; therefore the angles $CEA, AED$ are equal to the angles $AED, DEB$. [Post. 4 and C.N. 1]

Let the angle $AED$ be subtracted from each; therefore the remaining angle $CEA$ is equal to the remaining angle $BED$. [C.N. 3]

Similarly it can be proved that the angles $CEB, DEA$ are also equal. Therefore etc. Q.E.D.

The updated proof of the Vertical Angle Theorem given below illustrates how the numbering of angles in a diagram can make a proof much easier to follow, especially when there are a good number of angles of interest in a proof.

**Proof 2.** Let $AB$ and $CD$ intersect at a point $E$ and label the angles as shown in the figure.

Since $CE$ stands on $AB$, by proposition 13, we have $\angle 1 + \angle 2 = 2$ right angles.
Similarly, since $AE$ stands on $CD$, we have $\angle 2 + \angle 3 = 2$ right angles. By common notion 1, $\angle 1 + \angle 2 = \angle 2 + \angle 3$. Subtracting $\angle 2$ from both sides, we get $\angle 1 = \angle 3$ as desired.

Similarly, it can be shown that $\angle 2 = \angle 4$.

Notice that the sum of all four angles meeting at point $E$ in Proposition I.15 is four right angles. This is true in general, and Proclus stated it as a corollary to this proposition. It is included as an exercise for the reader. Before we begin the exercises, we introduce the following definition.

**Definition 3.8.** In any triangle, a line segment starting from a vertex and meeting the line defined by its opposite side perpendicularly is an **altitude** of the triangle.

**Exercises 3.2**

1. Give a construction and corresponding proof for each of the listed propositions. Unlike Euclid’s versions for these propositions, make sure that each proof only relies on Propositions I.1–I.8. Be sure to justify each step and include helpful diagrams as needed.

   (a) Proposition I.9  
   (b) Proposition I.10  
   (c) Proposition I.11  
   (d) Proposition I.12

2. Give an updated version of Euclid’s proof of each of the listed propositions. Be sure to justify each step, substitute mathematical symbols where appropriate, and include helpful diagrams as needed.

   (a) Proposition I.13  
   (b) Proposition I.14

3. Prove Theorem 3.5 by completing each direction of the biconditional as given separately in the following two parts.

   (a) Prove that every point on the perpendicular bisector of a line segment $AB$ is equidistant from the segment’s endpoints.
   (b) Prove that every point that is equidistant from the endpoints of a line segment $AB$ lies on its perpendicular bisector.

4. Prove that the median to the base of an isosceles triangle is perpendicular to the base and bisects the opposite angle.
3.3 Propositions I.16 through I.28 and I.31

The next six propositions are all inequalities relating angle measures or segment lengths. Proposition I.16 is often called the “Exterior Angle Theorem” as it compares interior angles of a triangle with an angle exterior to the triangle. Proposition I.17 asserts that the sum of any two interior angles in a triangle is less than two right angles. This proposition will seem rather weak after Proposition I.32, the proposition every high school geometry student remembers: the sum of the angles in any triangle is equal to two right angles. Unlike Proposition I.32, Proposition I.17 belongs to Neutral geometry since its proof does not rely on the fifth postulate. Propositions I.18 and I.19 relate angle magnitude with side length and are converses of each other.

**Proposition I.16 [Exterior Angle Theorem].** *In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.*
Figure 3.26. Proposition I.16

Proof. Let $ABC$ be a triangle, and let one side of it $BC$ be produced to $D$;

I say that the exterior angle $ACD$ is greater than either of the interior and opposite angles $CBA, BAC$.

Let $AC$ be bisected at $E$ [I.10], and let $BE$ be joined and produced in a straight line to $F$;

let $EF$ be made equal to $BE$ [I.3], let $FC$ be joined [Post. 1], and let $AC$ be drawn through to $G$ [Post. 2].

Then, since $AE$ is equal to $EC$, and $BE$ to $EF$, the two sides $AE, EB$ are equal to the two sides $CE, EF$ respectively; and the angle $AEB$ is equal to the angle $FEC$, for they are vertical angles. [I.15]

Therefore the base $AB$ is equal to the base $FC$, and the triangle $ABE$ is equal to the triangle $CFE$, and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; [I.4]

therefore the angle $BAE$ is equal to the angle $ECF$.

But the angle $ECD$ is greater than the angle $ECF$; [C.N. 5] therefore the angle $ACD$ is greater than the angle $BAE$.

Similarly also, if $BC$ be bisected, the angle $BCG$, that is, the angle $ACD$ [I.15], can be proved greater than the angle $ABC$ as well.

Therefore etc. Q.E.D. 

The proof of the following related theorem, which uses Proposition I.16 to establish the uniqueness of a perpendicular through a point, is left as an exercise.

Theorem 3.9. Let $C$ be a point not on the line $\overrightarrow{AB}$. There is exactly one line through $C$ that is perpendicular to $\overrightarrow{AB}$.

Proposition I.17. In any triangle two angles taken together in any manner are less than two right angles.

Proof. Let $ABC$ be a triangle; I say that two angles of the triangle $ABC$ taken together in any manner are less than two right angles. For let $BC$ be produced to $D$. [Post. 2]
Then, since the angle $ACD$ is an exterior angle of the triangle $ABC$, it is greater than the interior and opposite angle $ABC$. [I.16]

Let the angle $ACB$ be added to each; therefore the angles $ACD, ACB$ are greater than the angles $ABC, BCA$.

But the angles $ACD, ACB$ are equal to two right angles. [I.13]

Therefore the angles $ABC, BCA$ are less than two right angles.

Similarly we can prove that the angles $BAC, ACB$ are also less than two right angles, and so are the angles $CAB, ABC$ as well.

Therefore etc. Q.E.D.

**Proposition I.18.** In any triangle the greater side subtends the greater angle.

**Proof.** For let $ABC$ be a triangle having the side $AC$ greater than $AB$;

I say that the angle $ABC$ is also greater than the angle $BCA$.

For, since $AC$ is greater than $AB$, let $AD$ be made equal to $AB$ [I.3], and let $BD$ be joined.

Then, since the angle $ADB$ is an exterior angle of the triangle $BCD$, it is greater than the interior and opposite angle $DCB$. [I.16]

But the angle $ADB$ is equal to the angle $ABD$, since the side $AB$ is equal to $AD$; therefore the angle $ABD$ is also greater than the angle $ACB$;

therefore the angle $ABC$ is much greater than the angle $ACB$.

Therefore etc. Q.E.D.
Proposition I.19. In any triangle the greater angle is subtended by the greater side.

Proof. Let \( \triangle ABC \) be a triangle having the angle \( \angle ABC \) greater than the angle \( \angle BCA \); I say that the side \( AC \) is also greater than the side \( AB \).

For, if not, \( AC \) is either equal to \( AB \) or less.

Now \( AC \) is not equal to \( AB \); for then the angle \( \angle ABC \) would also have been equal to the angle \( \angle ACB \); [I.5] but it is not; therefore \( AC \) is not equal to \( AB \).

Neither is \( AC \) less than \( AB \), for then the angle \( \angle ABC \) would also have been less than the angle \( \angle ACB \); [I.18] but it is not; therefore \( AC \) is not less than \( AB \).

And it was proved that it is not equal either.

Therefore \( AC \) is greater than \( AB \).

Therefore etc. Q.E.D.

Definition 3.10. The distance from a point \( A \) to a line \( \overrightarrow{BC} \) is defined to be the length of the shortest segment \( AD \) where \( D \) is any point on \( BC \).

We leave the proof of the following theorem as Exercise 3.3.7.
**Theorem 3.11.** Of all line segments joining a point not on a given line to the line, the unique shortest segment is perpendicular to the given line.

Known as the Triangle Inequality Theorem, Proposition I.20 asserts that the sum of the lengths of two sides of a triangle is greater than the third side. According to Proclus, the Epicureans of Greece ridiculed the inclusion of this proposition in the *Elements* since it is “evident even to an ass and requiring no proof” [40]. They claimed that if you were to place a donkey at one vertex of a triangle and his food at another, he would never traverse the two sides over the one to get to it. Proclus’ response was, “that a mere perception of the truth of a theorem is a different thing from a proof of it and a knowledge of why it is true” [40]. We add to Proclus’ wisdom with the observation that it is often the most obvious that is most difficult to prove.

**Proposition I.20 [Triangle Inequality Theorem].** In any triangle two sides taken together in any manner are greater than the remaining one.

![Figure 3.31. Proposition I.20](image)

**Proof.** For let $ABC$ be a triangle; I say that in the triangle $ABC$ two sides taken together in any manner are greater than the remaining one, namely $BA, AC$ greater than $BC$, $AB, BC$ greater than $AC$, $BC, CA$ greater than $AB$.

For let $BA$ be drawn through to the point $D$, let $DA$ be made equal to $CA$, and let $DC$ be joined.

Then, since $DA$ is equal to $AC$, the angle $ADC$ is also equal to the angle $ACD$; [I.5] therefore the angle $BCD$ is greater than the angle $ADC$. [C.N. 5]

And, since $DCB$ is a triangle having the angle $BCD$ greater than the angle $BDC$, and the greater angle is subtended by the greater side, [I.19] therefore $DB$ is greater than $BC$.

But $DA$ is equal to $AC$; therefore $BA$, $AC$ are greater than $BC$.

Similarly we can prove that $AB$, $BC$ are also greater than $CA$, and $BC$, $CA$ than $AB$. Therefore etc. Q.E.D.

**Proposition I.21.** If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.
Chapter 3 Book I: Neutral Geometry

Figure 3.32. Proposition I.21

Proof. On \( BC \), one of the sides of the triangle \( ABC \), from its extremities \( B, C \), let the two straight lines \( BD, DC \) be constructed meeting within the triangle;

I say that \( BD, DC \) are less than the remaining two sides of the triangle \( BA, AC \), but contain an angle \( BDC \) greater than the angle \( BAC \).

For let \( BD \) be drawn through to \( E \).

Then, since in any triangle two sides are greater than the remaining one, \([I.20]\) therefore, in the triangle \( ABE \), the two sides \( AB, AE \) are greater than \( BE \).

Let \( EC \) be added to each; therefore \( BA, AC \) are greater than \( BE, EC \).

Again, since, in the triangle \( CED \), the two sides \( CE, ED \) are greater than \( CD \), let \( DB \) be added to each; therefore \( CE, EB \) are greater than \( CD, DB \).

But \( BA, AC \) were proved greater than \( BE, EC \); therefore \( BA, AC \) are much greater than \( BD, DC \).

Again, since in any triangle the exterior angle is greater than the interior and opposite angle, \([I.16]\) therefore, in the triangle \( CDE \), the exterior angle \( BDC \) is greater than the angle \( CED \).

For the same reason, moreover, in the triangle \( ABE \) also, the exterior angle \( CEB \) is greater than the angle \( BAC \). But the angle \( BDC \) was proved greater than the angle \( CEB \); therefore the angle \( BDC \) is much greater than the angle \( BAC \).

Therefore etc. Q.E.D.

In Proposition I.22, Euclid constructs a triangle from three segments where the sum of any two is greater than the third. Euclid’s statement of I.22 incorporates his conclusion from I.20, which may cause confusion. In the “thus it is necessary that...” clause of this proposition, Euclid includes the requirement from I.20 that the sum of any two sides of a triangle must be greater than the third. In other words, if we were given three line segments that fail to meet this condition, say segments of length 1, 1 and 3, then it would be impossible to construct a triangle with these lengths. To be clear, Euclid’s secondary clause in the proposition could be folded into the hypothesis of the statement as follows: 

*Given three straight lines where two of the straight lines taken together in any manner should be greater than the remaining one, to construct a triangle.*

We take this opportunity to note that there are many equivalent ways of rephrasing the conditional statement “If \( P \), then \( Q \).” These include “\( Q \), if \( P \);” “\( P \) only if \( Q \);” “\( P \) is a
sufficient condition for \( Q \); and “\( Q \) is a necessary condition for \( P \)”.
So, while in I.20 Euclid shows that “the sum of any two sides greater than the third” is a necessary condition for a triangle, here he shows that it is also a sufficient condition. This in turn allows Euclid to copy any triangle and hence, as we will see in Proposition I.23, any angle.

**Proposition I.22.** Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I.20]

![Figure 3.33. Proposition I.22](image)

**Proof.** Let the three given straight lines be \( A, B, C \), and of these let two taken together in any manner be greater than the remaining one, namely \( A, B \) greater than \( C \), \( A, C \) greater than \( B \), and \( B, C \) greater than \( A \); thus it is required to construct a triangle out of straight lines equal to \( A, B, C \).

Let there be set out a straight line \( DE \), terminated at \( D \) but of infinite length in the direction of \( E \), and let \( DF \) be made equal to \( A \), \( FG \) equal to \( B \), and \( GH \) equal to \( C \). [I.3]

With centre \( F \) and distance \( FD \) let the circle \( DKL \) be described; again, with centre \( G \) and distance \( GH \) let the circle \( LKH \) be described; and let \( KF, KG \) be joined;

I say that the triangle \( KFG \) has been constructed out of three straight lines equal to \( A, B, C \).

For, since the point \( F \) is the centre of the circle \( DKL \), \( FD \) is equal to \( FK \).
But \( FD \) is equal to \( A \); therefore \( KF \) is also equal to \( A \).

Again, since the point \( G \) is the centre of the circle \( LKH \), \( GH \) is equal to \( GK \).
But \( GH \) is equal to \( C \); therefore \( KG \) is also equal to \( C \). And \( FG \) is also equal to \( B \); therefore the three straight lines \( KF, FG, GK \) are equal to the three straight lines \( A, B, C \).

Therefore out of the three straight lines \( KF, FG, GK \), which are equal to the three given straight lines \( A, B, C \), the triangle \( KFG \) has been constructed.

Q.E.F. \( \square \)
A careful reader will notice that Euclid does not justify the intersection of the two constructed circles. This is the same unstated assumption we found in the proof of Proposition I.1.

Proposition I.23 is the theorem that allows Euclid to copy an angle. However, Euclid’s reliance on Proposition I.22 makes his proof particularly unhelpful when viewed as a set of instructions. As an exercise, the reader is asked to give a more succinct set of instructions for this construction and provide the corresponding proof.

**Proposition I.23.** On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.

![Figure 3.34. Proposition I.23](image)

**Proof.** Let $AB$ be the given straight line, $A$ the point on it, and the angle $DCE$ the given rectilineal angle;

thus it is required to construct on the given straight line $AB$, and at the point $A$ on it, a rectilineal angle equal to the given rectilineal angle $DCE$.

On the straight lines $CD$, $CE$ respectively let the points $D$, $E$ be taken at random; let $DE$ be joined, and out of three straight lines which are equal to the three straight lines $CD$, $DE$, $CE$ let the triangle $AFG$ be constructed in such a way that $CD$ is equal to $AF$, $CE$ to $AG$, and further $DE$ to $FG$.

Then, since the two sides $DC$, $CE$ are equal to the two sides $FA$, $AG$ respectively, and the base $DE$ is equal to the base $FG$, the angle $DCE$ is equal to the angle $FAG$. [I.8]

Therefore on the given straight line $AB$, and at the point $A$ on it, the rectilineal angle $FAG$ has been constructed equal to the given rectilineal angle $DCE$.

Q.E.F.

These next two propositions provide inequalities relating angle magnitude with side length. They are similar in nature to Propositions I.18 and I.19, but while the earlier propositions considered only one triangle, these compare two.

**Proposition I.24.** If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.
Proposition I.24

Proof. Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$ respectively, namely $AB$ to $DE$, and $AC$ to $DF$, and let the angle at $A$ be greater than the angle at $D$;

I say that the base $BC$ is also greater than the base $EF$.

For, since the angle $BAC$ is greater than the angle $EDF$, let there be constructed, on the straight line $DE$, and at the point $D$ on it, the angle $EDG$ equal to the angle $BAC$; [I.23] let $DG$ be made equal to either of the two straight lines $AC$, $DF$, and let $EG$, $FG$ be joined.

Then, since $AB$ is equal to $DE$, and $AC$ to $DG$, the two sides $BA$, $AC$ are equal to the two sides $ED$, $DG$, respectively; and the angle $BAC$ is equal to the angle $EDG$; therefore the base $BC$ is equal to the base $EG$. [I.4]

Again, since $DF$ is equal to $DG$, the angle $DGF$ is also equal to the angle $DFG$; [I.5] therefore the angle $DFG$ is greater than the angle $EGF$.

Therefore the angle $EFG$ is much greater than the angle $EGF$.

And, since $EFG$ is a triangle having the angle $EFG$ greater than the angle $EGF$, and the greater angle is subtended by the greater side, [I.19] the side $EG$ is also greater than $EF$.

But $EG$ is equal to $BC$. Therefore $BC$ is also greater than $EF$.

Therefore etc. Q.E.D.

Proposition I.25. If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

Proof. Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$ respectively, namely $AB$ to $DE$, and $AC$ to $DF$; and let the base $BC$ be greater than the base $EF$;

I say that the angle $BAC$ is also greater than the angle $EDF$.

For, if not, it is either equal to it or less.

Now the angle $BAC$ is not equal to the angle $EDF$; for then the base $BC$ would also have been equal to the base $EF$, [I.4] but it is not; therefore the angle $BAC$ is not equal to the angle $EDF$.
Neither again is the angle $BAC$ less than the angle $EDF$; for then the base $BC$ would also have been less than the base $EF$, [I.24] but it is not; therefore the angle $BAC$ is not less than the angle $EDF$.

But it was proved that it is not equal either; therefore the angle $BAC$ is greater than the angle $EDF$.

Therefore etc. Q.E.D.

With the completion of the following proposition we will have proven the four congruence schemes for triangles: SAS, SSS, ASA and AAS. In terms of the possible different arrangements of letters when dealing with the three angles (A) or three sides (S) of a triangle, the only two missing possibilities are side-side-angle and angle-angle-angle. These are not, however, valid congruence schemes. It is easy to see that angle-angle-angle does not guarantee congruence by imaging a triangle along with a “pinch to zoom” enlargement of the same triangle. The smaller and larger versions share the same angles but clearly have different side lengths and are thus, not congruent. We ask the reader to produce a counterexample to side-side-angle as an exercise.

**Proposition I.26 [ASA], [AAS].** If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.

**Proof.** Let $ABC, DEF$ be two triangles having the two angles $ABC, BCA$ equal to the two angles $DEF, EFD$ respectively, namely the angle $ABC$ to the angle $DEF$, and the
angle $BCA$ to the angle $EFD$; and let them also have one side equal to one side, first that adjoining the equal angles, namely $BC$ to $EF$;

I say that they will also have the remaining sides equal to the remaining sides respectively, namely $AB$ to $DE$ and $AC$ to $DF$, and the remaining angle to the remaining angle, namely the angle $BAC$ to the angle $EDF$.

For, if $AB$ is unequal to $DE$, one of them is greater.

Let $AB$ be greater, and let $BG$ be made equal to $DE$; and let $GC$ be joined.

Then, since $BG$ is equal to $DE$, and $BC$ to $EF$, the two sides $GB, BC$ are equal to the two sides $DE, EF$ respectively; and the angle $GBC$ is equal to the angle $DEF$; therefore the base $GC$ is equal to the base $DF$, and the triangle $GBC$ is equal to the triangle $DEF$, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I.4] therefore the angle $GCB$ is equal to the angle $DFE$.

But the angle $DFE$ is by hypothesis equal to the angle $BCA$; therefore the angle $BCG$ is equal to the angle $BCA$, the less to the greater: which is impossible.

Therefore $AB$ is not unequal to $DE$, and is therefore equal to it.

But $BC$ is also equal to $EF$; therefore the two sides $AB, BC$ are equal to the two sides $DE, EF$ respectively, and the angle $ABC$ is equal to the angle $DEF$; therefore the base $AC$ is equal to the base $DF$, and the remaining angle $BAC$ is equal to the remaining angle $EDF$. [I.4]

Again, let sides subtending equal angles be equal, as $AB$ to $DE$;

I say again that the remaining sides will be equal to the remaining sides, namely $AC$ to $DF$ and $BC$ to $EF$, and further the remaining angle $BAC$ is equal to the remaining angle $EDF$.

For, if $BC$ is unequal to $EF$, one of them is greater.

Let $BC$ be greater, if possible, and let $BH$ be made equal to $EF$; let $AH$ be joined.

Then, since $BH$ is equal to $EF$, and $AB$ to $DE$, the two sides $AB, BH$ are equal to the two sides $DE, EF$ respectively, and they contain equal angles; therefore the base $AH$ is equal to the base $DF$, and the triangle $ABH$ is equal to the triangle $DEF$, and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I.4] therefore the angle $BHA$ is equal to the angle $EFD$.

But the angle $EFD$ is equal to the angle $BCA$; therefore, in the triangle $AHC$, the exterior angle $BHA$ is equal to the interior and opposite angle $BCA$: which is impossible. [I.16]

Therefore $BC$ is not unequal to $EF$, and is therefore equal to it.

But $AB$ is also equal to $DE$; therefore the two sides $AB, BC$ are equal to the two sides $DE, EF$ respectively, and they contain equal angles; therefore the base $AC$ is equal to the base $DF$, the triangle $ABC$ equal to the triangle $DEF$, and the remaining angle $BAC$ equal to the remaining angle $EDF$. [I.4]

Therefore etc. Q.E.D.
While side-side-angle is not a valid congruence scheme for triangles in general, there are two special cases where it holds: when the congruent angles are either right or obtuse. In a right triangle, the sides adjacent to the right angle are commonly referred to as the legs, while the side opposite to the right angle is called the hypotenuse. Consequently, when the congruent angles of two triangles are right, the scheme is more commonly referred to as hypotenuse-leg, abbreviated HL. We state the theorem and begin the proof, but leave the remainder to the reader. We also leave it as an exercise for the reader to provide a proof for SSA in the case where the congruent angles are obtuse.

**Theorem 3.12 [HL].** If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the triangles are congruent.

**Proof.** Consider right triangles $\triangle ABC$ and $\triangle DEF$ with right angles $\angle B$ and $\angle E$, respectively. Assume that $AC = DF$ and $AB = DE$. Extend the ray $CB$ to a point $G$ such that $GB = EF$. Join $AG$... \(\square\)

Recalling from Theorem 3.11 that the distance between a line and a point not on it is given by the length of the unique perpendicular segment from the point to the line, we have the following theorem. The proof is left to the reader as Exercise 3.3.11.

**Theorem 3.13.** A point interior to an angle $\angle BAC$ lies on the angle’s bisector if and only if it is equidistant from the rays $\overline{AB}$ and $\overline{AC}$.

![Figure 3.38. Angles created by a transversal to parallel lines](image-url)

Propositions I.27 and I.28 are noteworthy in that they are the first two to mention parallel lines. Before we state the propositions, it would be helpful to explain some new terminology. When a pair of lines is crossed by another line, the crossing line is called a transversal. This arrangement of lines creates many angles of interest, of which several pairs are given identifying names. We will refer to Figure 3.38, which
shows lines $AB$ and $CD$ crossed by transversal $EF$, to define our new terms. Angles $\angle 1$, $\angle 4$, $\angle 6$ and $\angle 7$ are called **interior angles**. The pair of angles $\angle 1$ and $\angle 7$, or the pair $\angle 4$ and $\angle 6$, are called **alternate interior angles**. Angles $\angle 1$ and $\angle 6$, or angles $\angle 4$ and $\angle 7$, are interior and on the same side of the transversal. Angles $\angle 2$, $\angle 3$, $\angle 5$ and $\angle 8$ are called **exterior angles**. The pair of angles $\angle 2$ and $\angle 8$, or the pair $\angle 3$ and $\angle 5$, are called **alternate exterior angles**. Angles $\angle 2$ and $\angle 5$, or angles $\angle 3$ and $\angle 8$, are exterior and on the same side of the transversal. Angle $\angle 6$ is interior and **opposite** to the exterior angle $\angle 2$. The pair of angles $\angle 1$ and $\angle 2$, or the pair $\angle 2$ and $\angle 3$, are examples of supplementary, or **adjacent angles**. The pair of angles $\angle 1$ and $\angle 3$, or the pair $\angle 2$ and $\angle 4$, are examples of vertical angles.

Propositions I.27 and I.28 give conditions on these angles which guarantee that $AB$ and $CD$ are parallel. We write $AB \parallel CD$ when these lines are parallel.

**Proposition I.27.** If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

![Figure 3.39. Proposition I.27](image)

**Proof.** For let the straight line $EF$ falling on the two straight lines $AB, CD$ make the alternate angles $AEF, EFD$ equal to one another;

I say that $AB$ is parallel to $CD$.

For, if not, $AB, CD$ when produced will meet either in the direction of $B, D$ or towards $A, C$.

Let them be produced and meet, in the direction of $B, D$, at $G$.

Then, in the triangle $GEF$, the exterior angle $AEF$ is equal to the interior and opposite angle $EFG$: which is impossible. [I.16]

Therefore $AB, CD$ when produced will not meet in the direction of $B, D$.

Similarly it can be proved that neither will they meet towards $A, C$.

But straight lines which do not meet in either direction are parallel; [Def. 23] therefore $AB$ is parallel to $CD$.

Therefore etc. Q.E.D. 

Referring back to the numbered angles in Figure 3.38, Proposition I.27 states that if either $\angle 1 = \angle 7$ or $\angle 4 = \angle 6$, then $AB \parallel CD$. In the next proposition, we will show that if $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, $\angle 4 = \angle 8$, or $\angle 1 = \angle 5$, then $AB \parallel CD$. Moreover, if $\angle 1 + \angle 6$ or $\angle 4 + \angle 7$ equals two right angles, then $AB \parallel CD$. 
**Proposition I.28.** If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

**Proof.** For let the straight line EF falling on the two straight lines \(AB, CD\) make the exterior angle \(EGB\) equal to the interior and opposite angle \(GHD\), or the interior angles on the same side, namely \(BGH, GHD\), equal to two right angles;

I say that \(AB\) is parallel to \(CD\).

For, since the angle \(EGB\) is equal to the angle \(GHD\), while the angle \(EGB\) is equal to the angle \(AGH\), [I.15] the angle \(AGH\) is also equal to the angle \(GHD\); and they are alternate; therefore \(AB\) is parallel to \(CD\). [I.27]

Again, since the angles \(BGH, GHD\) are equal to two right angles, and the angles \(AGH, BGH\) are also equal to two right angles, [I.13] the angles \(AGH, BGH\) are equal to the angles \(BGH, GHD\).

Let the angle \(BGH\) be subtracted from each; therefore the remaining angle \(AGH\) is equal to the remaining angle \(GHD\); and they are alternate; therefore \(AB\) is parallel to \(CD\). [I.27]

Therefore etc. Q.E.D.

While the previous two propositions had our first references to parallel lines, neither of the proofs relied on Postulate 5, the Parallel Postulate. As such, these propositions belong to Neutral geometry. Finally, though it is out of numerical order, we include Proposition I.31 since it, too, belongs to Neutral geometry even though it concerns the construction of a parallel line. Once again, because of its reliance on Proposition I.23, the proof is not useful as a set of instructions for constructing the parallel line. As an exercise, you will be asked to give a more succinct set of steps and provide the corresponding proof.

**Proposition I.31.** Through a given point to draw a straight line parallel to a given straight line.
**Figure 3.41. Proposition I.31**

*Proof.* Let $A$ be the given point, and $BC$ the given straight line; thus it is required to draw through the point $A$ a straight line parallel to the straight line $BC$.

Let a point $D$ be taken at random on $BC$, and let $AD$ be joined; on the straight line $DA$, and at the point $A$ on it, let the angle $DAE$ be constructed equal to the angle $ADC$ [I.23]; and let the straight line $AF$ be produced in a straight line with $EA$.

Then, since the straight line $AD$ falling on the two straight lines $BC$, $EF$ has made the alternate angles $EAD$, $ADC$ equal to one another, therefore $EAF$ is parallel to $BC$. [I.27]

Therefore through the given point $A$ the straight line $EAF$ has been drawn parallel to the given straight line $BC$.

Q.E.F.

With the completion of this proof, we have finished the 29 propositions of Book I that do not rely on the Parallel Postulate. We previously classified these results under two general headings, either constructions or relationships between geometric objects. Now we can refine this classification. Just over a third of these propositions are constructions: triangles in 1 and 22; lines in 11, 12 and 31; bisections in 9 and 10; and copying in 2, 3 and 23. Six propositions establish the congruence of two objects: triangles in 4, 8 and 26; and angles or sides in 5, 6 and 15. [Note: Proposition I.7 is a lemma for I.8.] There are eight propositions comparing relative magnitudes of angles or sides: angles in 16 and 17; sides in 20; angles and sides in 18, 19, 21, 24 and 25. Two propositions establish when, and if, an object is a straight line: 13 and 14. Lastly, two propositions establish when lines are parallel: 27 and 28.

**Exercises 3.3**

1. Give an updated version of Euclid’s proof of each of the listed propositions. Be sure to justify each step, substitute mathematical symbols where appropriate, and include helpful diagrams as needed.

(a) Proposition I.16  
(b) Proposition I.17  
(c) Proposition I.18  
(d) Proposition I.19  
(e) Proposition I.20  
(f) Proposition I.21  
(g) Proposition I.22  
(h) Proposition I.24  
(i) Proposition I.25  
(j) Proposition I.26  
(k) Proposition I.27  
(l) Proposition I.28
2. Whereupon \( \angle ACB \) is “added to each,” Euclid’s proof of Proposition I.17 utilizes a nonexistent property of magnitudes. Give the missing property using inequality notation to justify this step of the proof.

3. Give a construction and corresponding proof of Proposition I.23 that only relies on Propositions I.1–I.8.

4. Give a construction and corresponding proof of Proposition I.31 that only relies on Propositions I.1–I.8 and Proposition I.27.

5. Consider the isosceles triangle \( \triangle ABC \) as shown in Figure 3.42. Assume that \( AB = AC \) and \( \angle 1 = \angle 2 \). Prove that \( \triangle ADE \) is also an isosceles triangle.

6. Prove Theorem 3.9: Through a given point \( C \) not on line \( AB \), there is only one straight line perpendicular to \( AB \).

7. Prove Theorem 3.11. To do this, first use the uniqueness of the perpendicular as in Exercise 6 (Theorem 3.9), then prove that the perpendicular is of shortest length. To do this, let \( C \) be a point not on line \( AB \), and let \( D \) be constructed on \( AB \) so that \( CD \perp AB \). Prove that if \( E \) is any other point on \( AB \), then \( CE > CD \).

8. Provide a counterexample to demonstrate why side-side-angle (SSA) is not a valid congruence scheme for triangles.

9. Finish the proof of Theorem 3.12 (HL). Include a diagram.

10. Prove the congruence scheme SSA in the case when the congruent angles are obtuse. That is, consider triangles \( \triangle ABC \) and \( \triangle DEF \), where \( AB = DE, BC = EF, \angle ACB = \angle DFE \), and \( \angle ACB \) is obtuse. Prove that \( \triangle ABC \cong \triangle DEF \). Include a diagram. [Hint: Using a proof by contradiction, WLOG, assume \( AC > DF \). Construct \( G \) on \( AC \) such that \( GC = DF \). Prove \( \triangle BGC \cong \triangle EDF \), and then consider isosceles triangle \( \triangle ABG \).]

11. Prove Theorem 3.13 by completing each direction of the biconditional as given separately in the following two parts.

(a) Prove that every point on an angle’s bisector is equidistant to that angle’s sides. That is, let \( AD \) be the angle bisector of \( \angle BAC \). Pick \( E \) on \( AD \). Construct perpendiculars \( EF \) and \( EG \) to \( AB \) and \( AC \), respectively, where \( F \) lies on \( AB \), and \( G \) lies on \( AC \). Prove that \( EF = EG \).
(b) Prove that every point interior to an angle and equidistant to its sides lies on the angle’s bisector. That is, let \( D \) lie on the interior of \( \angle BAC \). Construct perpendiculars \( DE \) and \( DF \) to \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \), respectively, where \( E \) lies on \( \overrightarrow{AB} \), and \( F \) lies on \( \overrightarrow{AC} \). If \( DE = DF \), prove that \( \overrightarrow{AD} \) bisects \( \angle BAC \).

12. Prove that if the diagonals of a convex quadrilateral bisect each other, then the quadrilateral is a parallelogram.

13. Prove that a rectangle is a parallelogram.

14. As given in Euclid’s twenty-second definition of Book I, a rhombus is an equilateral quadrilateral. Give a proof for each of the following statements related to a rhombus.
   (a) Prove that a rhombus is a parallelogram.
   (b) Prove that the diagonals of a rhombus intersect at right angles and bisect each other.
   (c) Prove that if the diagonals of a quadrilateral intersect at right angles and bisect each other, then the quadrilateral is a rhombus.

15. In \( \triangle ABC \), assume \( AB < AC \) and let \( D \) be the intersection of the angle bisectors at \( B \) and \( C \). Prove that \( DB < DC \).

16. In \( \triangle ABC \), assume \( AB < BC \) and let \( E \) be the midpoint of \( AC \). Prove that \( \angle CBE < \angle ABE \). [Hint: Extend \( BE \) to a point \( F \) such that \( BE = EF \).]

17. In \( \triangle ABC \), assume \( AB < BC \) and let the angle bisector of \( \angle ABC \) intersect \( AC \) at \( D \). Prove that \( AD < CD \). [Hint: Use Exercise 16.]

18. Consider triangle \( \triangle ABC \). Extend sides \( AB \) and \( AC \) to \( H \) and \( J \), respectively. Let the angle bisectors of exterior angles \( \angle CBH \) and \( \angle BCJ \) intersect at a point \( D \). From \( D \), construct perpendicular \( DE \) to \( \overrightarrow{AH} \), as illustrated in Figure 3.43. Prove that the length of \( AE \) is half the perimeter of \( \triangle ABC \). [Hint: From \( D \), construct perpendiculars \( DF \) and \( DG \) to \( \overrightarrow{AJ} \) and \( BC \), respectively.]

![Figure 3.43. Exercise 3.3.18: \( AE \) is half the perimeter of \( \triangle ABC \)](image)
Figure 3.44. Partial dependency graph for Neutral geometry propositions
Now that we have introduced Euclid’s definitions, axioms and common notions, and have spent some time exploring propositions from the first book of the *Elements*, let’s take a step back and look at a completely different geometry, *Spherical geometry*. In doing so, we shift our gaze from the familiar flat Euclidean world to a round, three-dimensional world ripe for exploration. While Spherical geometry may present a new mathematical world, it is in many ways as natural as Euclidean geometry. After all, as the NASA photograph reminds us, we live on a giant sphere (albeit, imperfect).

As a first task, we will determine how our main characters behave on this new surface. To do this, we must ask ourselves, “What do a line and a circle look like on
a sphere?" To help visualize and formulate an answer to this question, we encourage the reader to have a sphere in hand before reading any further. This is similar to using a piece of paper as a model for a Euclidean plane. There are many readily available models of spheres which can be used. A baseball or an orange will work in a pinch, but these are a bit too small. We prefer a Lénárt Sphere™, but realize it is pricey and hard to find. A volleyball or basketball has the proper dimensions, but since we will be writing on the sphere, an inexpensive plastic ball of roughly the same size is best as it works well with dry or wet-erase markers. It is also helpful to have a handful of rubber bands that fit around the sphere, a piece of string able to stretch halfway around the sphere, and some chalk, pencils, or markers to write (and hopefully, erase) on your spherical model.

4.1 What is a straight line, anyways? - Part 2

With sphere in hand, let’s try to build our intuition for working within Spherical geometry. The first thing we will do is determine the behavior of our main characters, with an initial focus on the circle since it is more natural to visualize this figure on a sphere. As in life, it’s often good to start with what you know, in this case, Euclidean geometry. Since we are quite skilled at constructing a circle on a plane, perhaps the same idea will translate to the sphere. Our definition of a circle is the set of all points equidistant from a given center. So, first mark a point on your spherical model which will be the center of our circle. Take a moment to think about how we could mark all of the points on the sphere that are the same distance from your chosen center. On a plane we can use a compass to form a circle, but if we did not have that tool handy, we could also use a pin, a piece of string, and a marker. With the pin placed at the center, the string tied to the pin, and the marker tied to the other end of the string at the desired length, a careful sweep of the marker over the plane, keeping the string taut, will produce a circle on the plane. This construction on the plane should be enough to convince you that we can produce a circle in a similar manner on the sphere.

Let’s construct a circle on our spherical model. Since a pin might puncture the sphere, we will use a pencil as a substitute, and a shorter pencil will be easier to manage. Tie one end of the string around the pencil in a tight knot on the metal band near the eraser, then loop the other end of the string around the marker in a loose knot so we can adjust the length of taut string between them. If your sphere is roughly the size of a soccer ball, then three inches is a good distance between the pencil and marker for an initial attempt. Place the eraser on the point marked as the center of the circle. Depending upon your level of dexterity, the next step might work better with two people. Have one person hold the pencil in place while the other sweeps out the circle with the marker, ensuring that the string remains taut at its fixed length. The resulting figure on the sphere is a circle. See Figure 4.2 for a representation. (Figure 4.2 and several other figures were made using Spherical Easel, a program written by David Austin and Will Dickinson for creating interactive diagrams in Spherical geometry. It can be downloaded at merganser.math.gvsu.edu/easel.) Take some time to draw a few circles of various length on your sphere. A compass, more precisely, a spherical compass, makes this procedure much easier, but this tool is quite specialized and probably not in your desk drawer. As you draw some circles, try to imagine the shape of this time-saving tool.
Let's look at some similarities and differences between circles in Euclidean geometry and circles on the sphere. The definition and construction method are essentially the same. When you hold the sphere up for inspection, you may notice that your circle looks a lot like a circle in the plane. There is, however, a major difference in the size of circles that can be produced in one geometry versus the other. On a plane there is no theoretical limit to the size of a circle. On a sphere, the largest possible circle is one that splits the sphere precisely into two equal halves, or hemispheres. If we imagine the earth as a perfect sphere, the equator (90° latitude) is such a circle. A circle of maximal size is called a great circle, and we define it to be the intersection of the sphere with a plane that passes through the center of the sphere. On the earth, in addition to the equator, the most well-known examples of great circles are the longitude lines. Notice that, with the exception of the equator, no latitude is a great circle. For example, when we imagine a plane cutting through the 66.5° S latitude line (roughly the Antarctic circle), we lop off the bottom of a globe, but certainly do not go through its center.

On a sphere, two points are said to be antipodal if they are diametrically opposed, that is, situated at opposite sides of a diameter of the sphere, like the north and south pole. Each point \( A \) on the sphere, has a unique antipodal point \( A' \). Furthermore, the antipodal point of \( A' \) is \( A \). Notice that great circles contain infinitely many pairs of antipodal points since a plane cutting through the center of a sphere contains infinitely many diameters of the sphere.

Let's go back to our spherical model again. Suppose we have a plastic ball for which every great circle is 72 centimeters in circumference. If we mark a point \( N \) for the north pole, we could construct a great circle at the “equator” by making our string 18 cm (one-quarter the circumference of a great circle) and sweeping out the circle with the marker after centering the pencil at \( N \). If we make the string 36 cm in length (half the circumference of a great circle) and center the pencil at \( N \), we find that it is impossible to sweep out a circle and we can only mark the point \( S \) at the south pole. If we make the string 20 cm in length and center the pencil at \( N \), then we produce a circle in the southern hemisphere. Notice that this same circle could have been produced by centering the pencil at the antipodal point, \( S \), and setting the string at 16 cm in length. This ability to express the same circle using two different centers separates spherical
circles from Euclidean circles. By utilizing antipodal points in this way, a string with a length of one-quarter the circumference of a great circle is sufficient to produce any circle on a sphere.

Now let’s consider lines in Spherical geometry. First take a moment to review both the Euclidean definition of line and your answer to the question about the meaning of “straight line” in Section 1.3. Do either of these offer any help in determining the meaning of a straight line on a sphere? Following our Euclidean intuition about circles and their construction has led us to their spherical counterparts, the former produced by a compass and the latter by a spherical compass. Is there a spherical ruler we can use to construct a line on our spherical model with ease? It’s more than likely that neither Euclid’s definition nor your answers to the “What is a straight line?” thought experiment are offering much help here.

In order to get moving in the proper direction, we need to make a distinction between being an internal or an external viewer of a geometric world.\(^1\) (If you watched Flatland: The Movie as directed in Exercise 1.3.3, then you have already spent some time thinking about this distinction. There is a sequel, Flatland 2: Sphereland, that may also prove helpful while learning to visualize this geometry [53].) Imagine a small creature, let’s say a robotic ant, living in a two-dimensional Euclidean world. The ant is only aware of the plane on which it crawls, and sees no third dimension. The ant can crawl forward by moving its left and right legs forward in a highly precise robotic manner. There is no up or down. An external viewer lives outside this plane, say in a three-dimensional world, and can watch the ant crawl around on its plane. Let’s take an internal view; suppose you are the ant and the plane is solid white with no markings. How can you know when you are travelling straight ahead? Take a few minutes to think about what “straight” means for you as the ant. Remember, since you are a robotic ant your movement is precisely programmed. It may help to consider what happens when the ant is turning, that is, not travelling in a line. In a manner similar to a marching band holding formation through a turn, to proceed in a rightwards direction, the ant’s left legs travel farther than its right legs. If the right legs travel farther than the left legs, then the ant is turning left. If the right and left legs travel the same distance, then the ant is headed straight ahead.

Now let’s suppose the ant lives on a sphere roughly the size of our spherical model. Analogously, here the ant is only aware of the surface of the sphere on which it crawls, and there are no markings on the sphere. How can the ant know when it is travelling in a straight line? Suppose, for example, an external viewer estimates that the ant is walking along the 75°S latitude line of the sphere, which is close to the south pole. In order to maintain that path, the ant has to move the legs which are closer to the equator farther than the legs which are closer to the south pole. Alternatively, if the ant were crawling on the equator then both sets of legs travel the same distance. More generally, unless the ant is crawling on a great circle, one set of legs is travelling farther than the other. Thus, by the same reasoning we applied to the plane, this means that a great circle is a straight line on a sphere. Could that be correct? A line on a sphere is a circle!?

For further convincing, here are three experiments in increasing order of complexity and difficulty. First, try placing an appropriately sized rubber band on your spherical model. Where does it have to be placed so that it rests without shooting off

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\(^1\)As in Chapter 1, we take inspiration for this line of questioning from David Henderson [70].
4.1 What is a straight line, anyways? - Part 2

and taking out somebody’s eye? Second, after locating or creating a shallow puddle of water, roll a tennis ball very quickly through the puddle. When the ball travels through the puddle at a good speed then its path on the ground will be a straight line. What is the shape of the water mark transferred from the puddle to the tennis ball? Finally, only those of you who are circus performers should try this last experiment. In a pool, stand on a floating sphere. To cause the least harm to yourself and others when you fall (and you, most certainly, will fall), maintain a safe distance of at least twenty feet from any other person or object. Start walking on the sphere. If your path is a great circle and you have perfect balance then you could be up there impressing your audience for quite some time.

Here we have it: **On a sphere, straight lines are the great circles.** Perhaps you still find this troubling since you cannot help but see that a great circle is curvy, not straight. This is true, but this is the observation of an external viewer who sees the sphere but does not live on its surface. To the surface dwellers, great circles are intrinsically straight. So does this mean our two main characters have been pared down to a single, multiple-personality headliner in Spherical geometry? No. While every line is a circle, every circle is not a line. That said, the circle clearly has the leading role in Spherical geometry.

Now that we have an intuitive understanding for lines on a sphere, let’s take a look at how they behave as compared to their counterparts in Euclidean geometry. Suppose we have two antipodal points, let’s say $N$ and $S$, thinking of the north and south poles. By simply considering the great circles corresponding to longitudinal lines, it’s easy to see that there are infinitely many lines through $N$ and $S$. Likewise, there are infinitely many lines through any pair of antipodal points. Suppose $N$ and $A$ are not antipodal. Just as every city on the planet is on exactly one longitudinal line, there is exactly one great circle which passes through $N$ and $A$. By this reasoning, there is exactly one line through any pair of distinct nonantipodal points.

How will we define a **spherical line segment**? Informally, in Euclidean geometry, we think of a line segment as a finite part of a line with exactly two distinct endpoints. Given any great circle on a sphere and distinct points $A$ and $B$ that lie on it, how do we define a line segment joining them? As illustrated by Figure 4.3, the Euclidean line segment notation, $AB$, is ambiguous here since there are two arcs that connect these points. Between nonantipodal points $A$ and $B$, there is a shorter arc through $C$ called the **minor arc**, and a longer arc through $D$ called the **major arc**. We will distinguish these arcs by including identifying points. In the figure, we see that the minor arc is $\widehat{ACB}$ and the major arc is $\widehat{ADB}$. In general, we write $\widehat{EFG}$ to denote the arc of the great circle between points $E$ and $G$ that passes through $F$. At first glance, both arcs seem to satisfy our informal notion of what a line segment is. Thus, arcs $\widehat{ACB}$ and $\widehat{ADB}$ prompt us to discuss the difference between a straight line joining two points and the shortest path between those two points. As mentioned in Chapter 1, the following statement is often taken as a definition in high school geometry: A straight line is the shortest path between two points. This is still true on a sphere, which is why airplanes travel great circle routes around the earth. In Spherical geometry, the shortest path between nonantipodal points $A$ and $B$ is the minor arc, namely $\widehat{ACB}$ in Figure 4.3. The major arc, $\widehat{ADB}$, illustrates that not every arc between $A$ and $B$ is the shortest path. To ensure unambiguous notation, when $A$ and $B$ are nonantipodal we let $AB$ denote the unique minor arc that joins them. When $A$ and $B$ are antipodal we will not use this
notation since there are infinitely many great circles joining them. Instead, we will be careful to specify a particular line, or arc joining the points by giving a third point on the arc, when necessary.

Let’s go back to our understanding of circles armed with this new notation. Recall that given two nonantipodal points \(A\) and \(B\), if \(A\) is the center of a circle and \(B\) is a point on the circle, then we will use the minor arc connecting them, \(AB\), as our radius, and we will refer to this circle as the circle with center \(A\) and radius \(AB\). Note also that this circle can be expressed as having center \(A'\) and radius \(A'B\), where \(A'\) is the unique antipodal point for \(A\).

Finally, we end this section by briefly discussing the difference between distance and length. Distance is measured between two objects, say points, and length is a measurement of one object, say an arc. A minor arc, for example, has a length less than half the circumference of a great circle. The distance between any two nonantipodal points on the sphere is the length of the minor arc joining them, and hence, is less than half the circumference of a great circle. The distance between antipodal points is exactly half the circumference of a great circle. So, the distance between any two points on the sphere is at most half the circumference of a great circle. A major arc has length greater than half the circumference of a great circle but less than the full circumference. Lastly, a great circle has a length equal to its circumference.

A careful reader will notice that we’ve paused just short of defining a line segment in this section. We prefer to wait until after we’ve turned our attention to one of the key characters in Euclid’s first book, the triangle.

**Exercises 4.1**

1. Use your spherical model to help visualize the shape of a tool that would allow you to draw lines and line segments on your model, namely, a spherical ruler. Describe this construction tool. A diagram corresponding to your description would be helpful.

2. Use your spherical model to help visualize the shape of a tool that would allow you to draw circles on your model, namely, a spherical compass. The radius adjustment should be easier and faster than untying and retying knots in your string. Describe

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\(^2\)In Chapter 5 we give a rigorous definition of distance, but for now, we informally say that the distance between two points is the length of the shortest path joining them.
4.2 Triangles in Spherical geometry

Before we can examine the axioms and propositions of Book I within Spherical geometry, we need to carefully define a triangle on a sphere. Euclid himself does not specifically define a “triangle,” but rather classifies trilateral figures as either equilateral, isosceles or scalene. The idea that a triangle is simply a three-sided figure turns out to be too broad a classification on the sphere. In order to be able to prove things about these figures, we would like triangles on a sphere to resemble their counterparts in the Euclidean plane. In particular, we would like to preserve as many congruence schemes as possible for triangles. This means that we must explore the sides and angles of trilateral figures in Spherical geometry.

First, we define angles the same way that we do in the plane; namely, angles are formed when two lines intersect. For example, as we will see in Section 4.4, the intersection of any longitude line with the equator creates four right angles. Next, due to the nature of the sphere, we must ensure that the sides of any triangle only intersect at its vertices. For example, consider the figure created by arcs $\hat{A}BG$, $\hat{A}CF$ and $\hat{F}DG$ in Figure 4.4. To make the figure a bit easier to identify, we have flattened it, giving a two-dimensional representation in Figure 4.5. Even though the figure has three sides, we do not want to call this trilateral figure a triangle since major arcs $\hat{A}BG$ and $\hat{A}CF$ intersect at $E$, which is not a vertex of this three-sided figure. So, we need to place restrictions on arcs specifying which are allowable as sides of a triangle. For now, we will just note that not every arc can be the side of a triangle.

Given the importance of SAS in Euclidean geometry, ideally, we would like it to hold in Spherical geometry. Let’s explore this on our spherical model. Make a point on
the sphere using a marker. We’ll refer to this point as the north pole, $A$. Starting at the
north pole, draw a minor arc that ends in the northern hemisphere, that is, it does not
extend past the equator. Call the point at the end of this arc $B$. Draw another minor
arc starting at the north pole and ending in the northern hemisphere at a point, $C$, that
is not antipodal to $B$. Consider the figure contained by sides $AB$ and $AC$, and angle
$\alpha$, as shown in Figure 4.6. We seem to have two choices for the third side, namely,
either $BC$ or major arc $\widehat{BDC}$, where $D$ is some point on the great circle through $B$ and
$C$ that is obscured from our view by the sphere. The figure created by choosing arc $BC$
resembles a triangle. Though odd to imagine, the trilateral figure created by choosing
$\widehat{BDC}$ includes the lower hemisphere, as well as the region resembling a triangle, since
it must contain angle $\alpha$. So, the figure created by choosing $BC$ will lie completely inside
the figure created by choosing $\widehat{BDC}$. If we were to call both trilateral regions “triangles”
then the congruence scheme SAS would clearly not hold. To avoid the problem created
by allowing $\widehat{BDC}$ to be side the of a triangle, we restrict the length of a side to be less
than half of the circumference of a great circle. A pleasant consequence is that this
restriction also resolves the issue of the self-intersecting trilateral.

Another new concern when working on the sphere is determining the interior of
a figure. In the Euclidean plane, a triangle divides the plane into two regions, one
4.2 Triangles in Spherical geometry

with finite and one with infinite area. We choose the interior of the triangle to be the region of finite area, and the exterior to be the region with infinite area. With a surface area of \(4\pi R^2\), there is no such choice on a sphere of radius \(R\) since both regions have finite area. Let’s revisit the minor arcs \(AB\), \(AC\) and \(BC\) from our previous example. These three arcs form a trilateral figure that actually resembles a triangle in the plane, though it’s a bit bloated. See Figure 4.7 for a representative diagram. Shade this figure using a marker. Notice that sides \(AB\), \(AC\) and \(BC\) create two regions, the shaded and the unshaded, both of which are trilateral with finite area. Only the smaller region, however, resembles a triangle. If we were to call both regions “triangles” then the congruence scheme SSS clearly would not hold as both trilaterals share the same sides, but one has angles \(\alpha\), \(\beta\), and \(\gamma\) radians, while the other has angles \(2\pi - \alpha\), \(2\pi - \beta\), and \(2\pi - \gamma\) radians, respectively. Since we’d like SSS to hold, we cannot allow both the shaded and unshaded trilateral regions to be considered triangles. So, we need to place restrictions on the allowable angles for a spherical triangle. As is the case for Euclidean triangles, we require that each angle be less than \(\pi\).

We specify these two restrictions in the following definition of a spherical triangle.

**Definition 4.1.** A spherical triangle is a trilateral figure in which
- Any angle must be less than two right angles.
- Any side must be less than half of the circumference of a great circle.

Such figures are often referred to as “small triangles,” but when we use the term triangle in Spherical geometry we will mean these figures. Notice that the first restriction determines the interior of a trilateral figure. The second condition ensures that the sides joining any two vertices are well-defined minor arcs. You may wonder why we don’t allow sides that are exactly half the circumference of a great circle. In Exercise 4.2.1, the reader is asked to consider whether this is even possible in a trilateral figure.

We are now finally ready to define a line segment.

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3In Chapter 7 we give an axiomatic treatment of area. Here, we appeal to our informal notion of area.
Definition 4.2. In Spherical geometry, when \( A \) and \( B \) are nonantipodal points, the **spherical line segment** \( AB \) is the shortest path, or minor arc, joining the points. If \( A \) and \( A' \) are antipodal points lying on great circle, \( \ell \), then each arc joining \( A \) and \( A' \) is a line segment.

Thus, while major arc \( \widehat{ADB} \) is part of the line that joins \( A \) and \( B \), we will not call it a line segment.

Now that we have an understanding of the behavior of our main characters in this new geometry, we turn our attention to interpreting Euclid’s axioms on a sphere.

Exercises 4.2
1. Let’s examine what happens if we allow the definition of a spherical triangle to include a side of length one-half the circumference of a great circle. Let \( N, S \) and \( E \) be distinct points where \( N \) and \( S \) are antipodal. Let \( A \) be any point not on segment \( \widehat{NES} \). Construct segments \( AN \) and \( AS \). Do \( AN, AS \) and \( \widehat{NES} \) form a trilateral figure?

4.3 Euclid’s axioms viewed in Spherical geometry

Starting with five algebraic Common Notions and five geometric Postulates, Euclid proves one proposition after another using well-established logical constructs. Having interpreted and defined the basic terms of circles, lines, angles and triangles on a sphere, we are now ready to consider Euclid’s axioms within our new spherical world. Since the Common Notions are independent of the shape of our world, we can see that these five axioms are still valid in our new geometry. The five Postulates, however, concern our main characters, and their behavior has changed in Spherical geometry. Are these geometric axioms still valid? Using our working understanding of lines and circles on spheres, let’s revisit Euclid’s five postulates to carefully determine whether each postulate holds as it is stated. As the first two postulates have the most nuanced interpretations, for these we also consider how Euclid uses these postulates in the proofs, but we limit the scope of our analysis to Neutral geometry since this will be sufficient for our investigations. Before reading further, take a few minutes to carefully consider each postulate to determine whether or not there is an interpretation which admits its validity in Spherical geometry for our analysis of the propositions of Neutral geometry.

Postulate 1: To draw a straight line from any point to any point.

Given two points \( A \) and \( B \), this postulate is used to create a line segment, \( AB \), that joins them. If \( A \) and \( B \) are nonantipodal points, then there is a unique great circle passing through them and the line segment \( AB \) is the unique minor arc on this circle between them. If \( A \) and \( B \) are antipodal points, there are infinitely many great circles that contain them, thus the segment between them is not uniquely defined. In either case, given any two points \( A \) and \( B \), there is at least one line segment that joins them. Hence, with a strict interpretation, this postulate holds in Spherical geometry. If this matches your interpretation of Postulate 1, then skip ahead to Postulate 2.

If you are reading this paragraph then either you are simply curious or our interpretation does not match yours, and we are certain that uniqueness is at issue. How do we know this? There is often a tacit understanding that, while not explicitly stated,
the line segment is unique. If Euclid’s postulate is interpreted to include uniqueness then this postulate fails in Spherical geometry. We concede that this alternative is a valid interpretation, but to paraphrase Calvin in the comic strip at the start of Chapter 2, we do not want you to be excused from further study. Instead, in the case of antipodal points, we ask whether the lack of uniqueness of the segment will present any problems for our analysis of the Neutral geometry propositions on a sphere? Upon examination of Euclid’s use of this postulate within his proofs, it is common for him to use this postulate to join two points with a line segment, but Proposition I.4 is the only proof where he assumes that the line will be unique. What exactly does he assume in I.4? In the proof, Euclid “applies” one triangle to another and, while he does not cite any postulate, he claims that given two vertices of a triangle, $E$ and $F$, there is only one line segment connecting them. In this case, $EF$ is a side of a triangle. In Spherical geometry, vertices of a triangle are necessarily nonantipodal, and thus, there is a unique line segment joining $E$ and $F$. Therefore, in spite of the fact that the line between any two points is not necessarily unique in Spherical geometry, it is reasonable to argue that Postulate 1 still holds, and, this interpretation is sufficient for analyzing the Neutral geometry propositions.

**Postulate 2: To produce a finite straight line continuously in a straight line.**

Euclid uses this postulate to extend a given line segment. It is clear that given a line segment joining two points, we can always extend this segment into a great circle that contains the two points. Moreover, this great circle is unique. The word that may give us pause is “continuously.” Here it is helpful to distinguish between the notions of infinite and boundaryless. While a great circle is certainly finite in length, it is also free of any boundaries, in that we can continuously produce the line, never reaching an end. Let’s appeal to the distinction between the extrinsic and intrinsic views for greater clarity. As sailors quickly discovered, it is not possible to fall off the “edge of the world.” No such boundaries exist for the surface dwellers of a sphere. With this interpretation, it is reasonable to argue that this postulate holds in Spherical geometry. If this matches your interpretation of Postulate 2, then skip ahead to Postulate 3.

If you are reading this paragraph then our interpretation does not match yours, and we are certain that the finiteness of a great circle is at issue. If Euclid’s second postulate is interpreted to mean that a line has infinite length, then this postulate fails in Spherical geometry. We admit that there is one particular reference to this in Neutral geometry, specifically in the proof of Proposition I.22, which supports this view: “let there be set out a straight line $DE$ ... of infinite length...”. We concede that this alternative is a valid interpretation, but hear us out one more time. We ask again whether our interpretation of “continuously” will present any problems for our analysis of the validity of the Neutral geometry propositions on a sphere. Upon examination of Euclid’s use of this postulate within his proofs, it is clear that he intended for there to

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4In his book *A Survey of Geometry* [44], Howard Eves writes: “Although Postulate 2 asserts that a straight line may be produced indefinitely, it does not necessarily imply that a straight line is infinite in extent, but merely that it is endless, or boundless. The arc of a great circle joining two points on a sphere may be produced indefinitely along the great circle, making the prolonged arc endless, but certainly it is not infinite in extent. Now it is conceivable that a straight line may behave similarly, that after a finite prolongation it, too, may return on itself. It was Bernhard Riemann who in his famous dissertation, *Über die Hypothesen welche der Geometrie zu Grunde liegen*, of 1854, distinguished between the boundlessness and the infinitude of straight lines.”
be only one way to extend a line segment, and in this regard, our interpretation is on solid ground. Only occasionally does Euclid require that a given line be extended by a particular length. On the sphere, all but two of these cases translate to extending a minor arc of a line by at most a segment equal to a given side of a triangle.\footnote{The lone exceptions occur in Euclid’s proof of I.2, which we handle carefully in the next section by adding it to our Spherical axioms, and in his proof of I.22, which requires adding the lengths of all three sides of a triangle, a sum we show to be less than the circumference of a great circle.} Euclid’s language in I.22 spoke of the infinite but the word went far beyond his need. With our restriction on the length of a side of a spherical triangle, we will still be able to extend our lines as needed on a sphere. Therefore, in spite of the fact that producing a line continuously on a sphere means going around and around the same great circle, it is reasonable to argue that Postulate 2 holds for our purposes, that is, this interpretation is sufficient for analyzing the Neutral geometry propositions on a sphere.

**Postulate 3:** To describe a circle with any center and distance.

The word “distance” may at first seem troubling, but as discussed in Section 4.1, the distance between two points is the length of the shortest path joining them. For nonantipodal points, this is the length of the unique minor arc between them, and for antipodal points this is half the circumference of a great circle. So, distance can be at most $\pi R$ on a sphere of radius $R$. Given point $A$ and segment $AB$, using our compass we can construct a circle with the given center and radius since $AB$ refers to a minor arc of a great circle. In the maximal case where we are given point $C$ and segment $\hat{C}D\hat{C}'$ where $\hat{C}'$ is antipodal to $C$, the circle with center $C$ and radius $\hat{C}D\hat{C}'$ will be a point, namely $C'$. Therefore, we can describe a circle with any center and any possible distance. Thus, it is reasonable to argue that Postulate 3 holds.

**Postulate 4:** That all right angles are equal to one another.

By the symmetry of the sphere, Postulate 4 holds.

**Postulate 5:** That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

To consider whether this postulate holds, we must first consider two straight lines in Spherical geometry. Without loss of generality, suppose that one of the lines is the equator, $e$. Now imagine any other great circle, $\ell$, on the sphere and you will see why any two distinct lines in Spherical geometry must intersect at two points. In fact, they will intersect at antipodal points, say $A$ and $A'$. (Try this on your spherical model!) In Euclidean geometry, a line that crosses a pair of lines (but not at an intersection point of the lines) is called a transversal. Every line cuts the plane into two sides, and the Parallel Postulate specifies on which side of the transversal the pair of lines must meet. In Spherical geometry, every line cuts the sphere in two hemispheres, and the Parallel Postulate specifies in which hemisphere (relative to the transversal) the pair of lines must meet. Suppose $t$ is a transversal to lines $e$ and $m$. (Note that $t$ cannot meet $e$ and $\ell$ at their antipodal points of intersection, $A$ and $A'$, for the same reason a transversal in Euclidean geometry cannot meet the pair of lines where the lines intersect.) Since $e$ and $\ell$ meet at antipodal points and $t$ divides the sphere into two hemispheres, $A$ will be on one side and $A'$ on the other. Therefore, $e$ and $\ell$ meet on both sides of the transversal. Since $t$ is an arbitrary transversal, the implication is always true and Postulate 5 holds.
4.4 Neutral geometry on the sphere

With the given interpretations, all of Euclid’s axioms as they are written and used in Neutral geometry can be considered valid in Spherical geometry. In the next section we consider the validity of the propositions of Neutral geometry on the sphere.

4.4 Neutral geometry on the sphere

In Chapter 3 we proved the 29 propositions of Neutral geometry from Book I of the Elements assuming ten axioms (Well actually, we only need nine for Neutral geometry!) and utilizing well-established logical constructs. Since these ten axioms are valid in Spherical geometry under the interpretations given in the previous section, the same 29 propositions of Book I should be true on a sphere. Let’s take another look at the statements and proofs of these Neutral geometry propositions.

We’ll start with Proposition I.1: On a given finite straight line to construct an equilateral triangle. At first glance, this appears to hold exactly as it does in Euclidean geometry. However, if we pick a segment of length between one-third and one-half of the circumference of a great circle, something surprising happens. Try it for your­self on your spherical model, but as Figure 4.8 illustrates, the two circles we construct in the proof of Proposition I.1 will not intersect. As we mentioned in Chapter 3 after

![Figure 4.8. Two nonintersecting circles](image)

the proof of I.1, none of Euclid’s Common Notions or Postulates guarantees that the constructed circles must intersect. Therefore, there must be an unstated assumption at work that Euclid is utilizing in order to claim that these circles meet. While not our first encounter with this “missing Euclidean postulate,” the sphere has given us a new way to identify this omission by Euclid, and perhaps, it will reveal others. For now, we will delay discussion of further implications of this observation in favor of merely noting that, in Spherical geometry, Proposition I.1 holds only if the segment is small enough, specifically, less than one-third of the circumference of a great circle. This restriction places a strict limit on the size of equilateral triangles that can be constructed on the sphere. We replace Euclid’s proposition with Proposition I.1S given below with the modification in boldface type, but we will delay its justification until the end of Section 4.6.

Proposition I.1S. On a given finite straight line less than one-third of the circumference of a great circle, to construct an equilateral triangle.
Let’s consider Proposition I.2: *To place at a given point [as an extremity] a straight line equal to a given straight line.* This is the proposition that allows us to assume that we have a rigid compass. Since the proof of this proposition requires the construction of an equilateral triangle on a segment of arbitrary length, we are in trouble. (The partial dependency tree for the propositions of Neutral geometry given in Figure 3.44 at the end of Chapter 3 will be helpful here.) Given the failure of I.1, clearly we cannot do this for segments of length at least one-third the circumference of a great circle. Euclid chose a collapsible compass as a construction tool in his postulates because he could fairly easily prove its equivalence to a rigid compass with his second proposition. We do not have this luxury in Spherical geometry, and thus, we have no choice but to revise our postulates to assume that our compass is rigid from the start.

**Spherical Postulate 6.** In Spherical geometry, the compass is rigid.

With the adoption of a rigid compass, both the statement and proof of Proposition I.3 (*Given two unequal straight lines, to cut off from the greater a straight line equal to the less.*) still hold. Given our definition of spherical triangles, Proposition I.4 (*If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.*) holds as well, and we have SAS for spherical triangles.

We will leave the investigation of the statements and proofs of Propositions I.5 through I.8 as exercises and will jump to Proposition I.9: *To bisect a given rectilineal angle.* In the proof of this proposition, it is crucial to notice that we have control over the length of segment $DE$. By picking $D$ close enough to $A$, we can ensure that equilateral triangle $\triangle DEF$ can be constructed. Thus, the proof holds with only one minor modification. While we are typically only interested in bisecting angles less than $\pi$, we leave it to the reader to show that, as is the case in Euclidean geometry, the constructed line $AF$ will also bisect the angle that is greater than $\pi$. Modifying the proof of Proposition I.10 (*To bisect a given finite straight line.*) proves to be a bit trickier. While we claim that the proposition still holds, Euclid’s proof will only work for segments that are less than one-third of the circumference of a great circle. We will defer a new, more general proof of this proposition until after our consideration of Proposition I.11 (*To draw a straight line at right angles to a given straight line from a given point on it.*) and Proposition I.13 (*If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.*)). Since the proofs of Proposition I.11 and Proposition I.13 do not use Proposition I.10, we are free to reorder them without fear of circular reasoning. We claim that Proposition I.11 and its proof still hold because, as in Proposition I.9, we are free to pick $D$ close to $C$ so that the equilateral triangle $\triangle FDE$ can be constructed. Proposition I.13 and its proof, which relies on Proposition I.11, will also still hold. Before we proceed with the proof of Proposition I.10, we define the **polar points**, or **poles**, of a line.

**Definition 4.3.** Let $A$ and $B$ be any two nonantipodal points on an arbitrary line in Spherical geometry. Construct perpendicular lines to $AB$ at points $A$ and $B$, and label the intersection points of these lines $C$ and $D$. Points $C$ and $D$ are called **polar points**, or **poles**, of the line containing $AB$. 

Lemma 4.4. The polar points for line $AB$ are antipodal points located a distance of one-quarter the circumference of the sphere from all points on the line $AB$.

We are now ready to give a new proof for Proposition I.10. We leave it to the reader to provide a diagram for each case.

Proposition I.10. To bisect a given finite straight line.

Proof. There are two cases depending upon whether or not the given segment is exactly half the circumference of a great circle in length.

Case 1. Suppose the endpoints of the given finite line are not antipodal. Let $AB$ be the line segment to be bisected. Construct a polar point $C$ of line $AB$. Join $AC$ and $BC$. By Lemma 4.4, we have $AC = BC$. Using I.9, bisect $\angle ACB$ and extend the line to point $E$ on $AB$. Then by Proposition I.4 (SAS), we have $\triangle ACE \cong \triangle BCE$. Thus, $AE = BE$ and we have bisected $AB$ at $E$.

Case 2. Suppose the endpoints of the given finite line are antipodal. Let $\overline{ABA}'$ be the given finite line where $B$ is some point between antipodal points $A$ and $A'$. Construct a polar point $C$ for segment $AB$. Join $AC$. Using I.11, construct a perpendicular to $AC$ at $C$. Let the intersection of this perpendicular with $\overline{ABA}'$ be $E$. We claim that $E$ bisects the given $\overline{ABA}'$. Since $C$ is a polar point to $AB$, and $B$ is on line $\overline{ABA}'$, by Lemma 4.4 we have $AC = A'C$. Furthermore, $AC$ can be extended to a line, and the antipodal point of $A$ is on that line. Therefore, $\overline{ACA}'$ is a straight line. As $\angle ACE$ is right by construction, then by I.13, angles $\angle ACE$ and $\angle A'CE$ are both right. Thus, by I.4 (SAS), $\triangle ACE \cong \triangle A'CE$, and hence $AE = A'E$ as desired.

We leave it to the reader to analyze Propositions I.12, I.14 and I.15 in a similar manner.

We now turn our attention to Proposition I.16 [Exterior Angle Theorem]: In any triangle, if one of the sides be produced, the exterior angle is greater than either of the
interior and opposite angles. As illustrated in Figure 4.10, we construct the spherical triangle $\triangle NBC$ by placing one vertex, $N$, at the North Pole and placing the two other vertices, $B$ and $C$, on the equator. We can see that the exterior angle $\angle NCD$ as well as the two interior angles $\angle NBC$ and $\angle NCB$ are all right angles. Thus the exterior angle $\angle NCD$ is equal to its opposite interior angle $\angle NBC$. Furthermore, we can choose $\angle BNC$ to be larger than a right angle. Thus, is it spectacularly clear that Proposition I.16 does not hold in Spherical geometry. We leave it as an exercise to investigate why the proof fails. We also leave the reader to analyze the validity of Proposition I.17 as we take a look at Propositions I.18 through I.20.

![Figure 4.10. A counterexample for Proposition I.16](image)

Euclid’s proof of Proposition I.18 (In any triangle the greater side subtends the greater angle.) relies on I.16. Continuing down this path, his proof of Proposition I.19 (In any triangle the greater angle is subtended by the greater side.) relies on Proposition I.18 and his proof of Proposition I.20 (In any triangle two sides taken together in any manner are greater than the remaining one.) relies on I.19. At first thought, this may lead us to question the validity of all three of these propositions in Spherical geometry since each relies on the failed I.16. Contrary to this initial speculation, on the sphere these propositions can be proven without calling on Proposition I.16.

As you may recall from Chapter 3, the Epicureans thought that the validity of Proposition I.20, also known as the Triangle Inequality, was obvious (even to an ass). This Epicurean observation holds in Spherical geometry as I.20 is valid, but we will not present the proof here since it relies on results from three-dimensional Euclidean geometry [112,117]. The important thing to note about the proof is that it relies on neither I.18 nor I.19. This means that we are free to use Proposition I.20 to prove either I.18 or I.19. We will prove Proposition I.19, but since the proof requires the copying of an angle we must once again venture a bit further ahead in the propositions.

We leave it to the reader to analyze the two parts of Proposition I.21. In Proposition I.22, a triangle is constructed from three segments where the sum of any two lengths is greater than the third. Unfortunately, not all such triangles are constructible on the sphere since it is possible to give three lengths for which the circles in the proof do not intersect. For example, $a = b = c = \frac{3\pi}{4}$ meets the condition of I.22, but a triangle with these sides is not constructible by I.1S. If, however, we also assume that the sum of the sides is less than the circumference of a great circle, then the triangle is constructible.
and the proof will hold. We replace Euclid’s proposition with Proposition I.22S given below with the modification in boldface type. We justify this modification with the following lemma.

**Lemma 4.5.** The total length of the sides of any spherical triangle on a sphere of radius $R$ is less than the circumference of a great circle, that is, $2\pi R$.

*Proof.* Consider a spherical triangle $\triangle ABC$. Extend $AB$ and $AC$ so that they meet at $A'$, the antipodal point of $A$. Then $\widehat{ABA'} + \widehat{ACA'} = 2\pi R$. Consider the spherical triangle $\triangle A'BC$. By I.20, we have $BC < BA' + CA'$. Thus, $AB + AC + BC < AB + AC + BA' + CA' = \widehat{ABA'} + \widehat{ACA'} = 2\pi R$. $\square$

**Proposition I.22S.** Given three straight lines whose sum is less than the circumference of a great circle, and such that the sum of any two is greater than the third, to construct a triangle.

This is good news for the proof of Proposition I.23 on the sphere since we can control the length of the segment required in the proof. Therefore, Proposition I.23 holds, and we can copy angles in Spherical geometry.

Now we are ready to go back to Proposition I.19 and present the proof from the 1871 text, *Spherical Trigonometry for the Use of Colleges and Schools*, by Isaac Todhunter (1820–1884) [117].

**Proposition I.19.** In any triangle the greater angle is subtended by the greater side.

![Figure 4.11. Proposition I.19](image)

*Proof.* Let $\triangle ABC$ be a spherical triangle, and let angle $\angle ABC$ be greater than angle $\angle BAC$. We wish to prove that side $AC$ will be greater than side $BC$. Let $D$ be the point on $AC$ such that angle $\angle ABD$ is equal to angle $\angle BAD$ [Proposition I.23]; then $BD$ is equal to $AD$ [Proposition I.6 on $\triangle ABD$], and $BD + DC$ is greater than $BC$ [Proposition I.20 on $\triangle BDC$]; therefore $AD + DC$ is greater than $BC$; that is, $AC$ is greater than $BC$. $\square$

We leave it to the reader to prove Proposition I.18 using Proposition I.19. We also leave the analysis of Propositions I.24 and I.25 as an exercise. This brings us to the final Euclidean triangle congruence scheme, AAS, which is found in the second half of Proposition I.26. The scheme AAS is not valid in Spherical geometry and we leave it to the reader to find a counterexample. Lastly, we leave it to the reader to analyze Propositions I.27, I.28 and I.31.
Let’s recap our findings and discuss the consequences. We established an interpretation for the validity of Euclid’s ten axioms for investigating the propositions of Neutral geometry on the sphere. Based on our work in Chapter 3 and our understanding of axiomatic systems, this means that these propositions should be true in both Euclidean and Spherical geometries since they start with the same axioms. However, we have clearly shown in this section that this is not the case since some of these propositions fail in Spherical geometry. So, where is the error in the reasoning? While one could argue that there are other possible interpretations of the postulates within Spherical geometry, we must acknowledge that the omissions we have highlighted, that is, assumptions made by Euclid that were not based on his axioms or previous propositions, are cause for concern. In other words, we know that Euclid’s set of ten axioms is incomplete and we will have to add some more. We are certainly missing axioms that distinguish Euclidean geometry from Spherical geometry, but we will postpone our augmentation of the axioms until after we consider another type of geometry in Chapter 5. For the remainder of this chapter, we investigate area, trigonometry and constructions in this strange, and yet familiar, new world.

**Elliptic and Double Elliptic geometries**

Spherical geometry is sometimes referred to as *Double Elliptic geometry*. The “double” refers to the fact that any two lines intersect in two distinct points. Felix Klein (1849-1925) created the closely related *Elliptic geometry* by identifying antipodal points on the sphere. That is, every antipodal pair, \( \{A, A'\} \), becomes one point. As a consequence, any two lines in Elliptic geometry intersect exactly once. Somewhat surprisingly, a line no longer separates the plane into two halves in this geometry. However, as is the case in Spherical Geometry, the Exterior Angle Theorem (I.16) does not hold.

**Exercises 4.4**

1. For Propositions I.5 through I.8 consider the following:
   - Does the proposition hold on a sphere?
   - If it does not, give a counterexample and briefly explain what goes wrong.
   - If it does hold, does Euclid’s proof work? If it does not, briefly explain what goes wrong. Note: you do not need to provide a valid proof.

2. Prove that the line \( AF \) constructed in the proof of Proposition I.9 will also bisect the angle that is greater than \( \pi \).

3. Prove Lemma 4.4: Show that the polar points for line \( AB \) are antipodal points located a distance of one-quarter the circumference of the sphere from all points on the line containing \( AB \).

4. Give a diagram to illustrate each case in the proof of Proposition I.10 in Spherical geometry.

5. Follow the directions given in Exercise 1 to analyze the validity of Propositions I.12, I.14 and I.15 in Spherical geometry.
6. Proposition I.16 does not hold in Spherical geometry. Consider the following triangle: place one vertex at the North Pole and place the two other vertices on the equator. Make the angle at the North Pole a right angle (or bigger). Carefully explain what goes wrong in Euclid’s proof of Proposition I.16.

7. Follow the directions given in Exercise 1 to analyze the validity of Proposition I.17 in Spherical geometry.

8. Consider a spherical triangle $\triangle ABC$. Show that $AB + BC \neq AC$ as if it were Proposition I.15. In your proof, be careful to use only the propositions that are valid in Spherical geometry.

9. Give a proof of Proposition I.18 that relies only on Proposition I.19 and propositions between I.4 and I.15 that are valid in Spherical geometry.

10. Follow the directions given in Exercise 1 to analyze the validity of Proposition I.21 in Spherical geometry. Be sure to analyze each of the two pieces separately.

11. Follow the directions given in Exercise 1 to analyze the validity of Propositions I.24 and I.25 in Spherical geometry.

12. Find a counterexample for AAS in Spherical geometry. That is, find two triangles $\triangle ABC$ and $\triangle DEF$ that have two pairs of congruent angles and a pair of congruent sides, but are not congruent triangles.

13. Follow the directions given in Exercise 1 to analyze the validity of Propositions I.27, I.28 and I.31 in Spherical geometry.

14. Give a proof of the following problem from a nineteenth century spherical trigonometry book: “If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle” [117].

4.5 Area in Spherical geometry

While we will postpone the strict axiomatic treatment of area until Chapter 7, we appeal to an informal notion of area to consider a very beautiful and somewhat unexpected result about the area of a spherical triangle. In order to prove this result, we must first introduce the spherical lune, a two-sided figure that does not exist in Euclidean geometry. As you read the definition, it may help to visualize the rind of a wedge of an orange, or to place two rubber bands on your spherical model.

**Definition 4.6.** Two distinct lines on a sphere divide the sphere into four regions, each of which is called a lune. Each lune has an angle, $\alpha$, which is given by the angle of intersection of the two lines.

Notice that if one of the lunes has angle $\alpha$ radians, the lune directly across the sphere from it will also have angle $\alpha$. The other pair of lunes will each have the supplementary angle $\pi - \alpha$ radians.

**Lemma 4.7.** On a sphere of radius $R$, the area of a lune with angle $\alpha$ radians is given by $Area = 2\alpha R^2$. 

A sphere of radius $R$ has surface area $4\pi R^2$. A lune of angle $\frac{\pi}{4}$ has one-quarter of the sphere’s total surface area, or $\pi R^2$, since it is a proportion of $\frac{\pi}{2\pi} = \frac{1}{4}$ of the total surface area. Thus, a lune of angle $\alpha$ must have a proportion of $\frac{\alpha}{2\pi}$ of the total surface area, or $\left(\frac{\alpha}{2\pi}\right)4\pi R^2 = 2\alpha R^2$.

We are now ready for our area result.

**Theorem 4.8.** On a sphere of radius $R$, the area of spherical triangle $\triangle ABC$ with angles $\alpha, \beta$ and $\gamma$ (measured in radians) is given by $\text{Area}(\triangle ABC) = (\alpha + \beta + \gamma - \pi)R^2$.

**Proof.** The sides of $\triangle ABC$ create lunes of angles $\alpha, \beta$ and $\gamma$, with areas of $2\alpha R^2$, $2\beta R^2$ and $2\gamma R^2$, respectively. Together, these three lunes cover exactly half of the area of the sphere, or $2\pi R^2$. (Use your spherical model to convince yourself of this by shading the three lunes.) Since each lune contains the triangle $\triangle ABC$, when we sum the area of
the three lunes we count the area of the triangle $\triangle ABC$ three times. Adjusting for this overcounting, we have

$$\text{sum of lunar areas} - 2\text{Area}(\triangle ABC) = \frac{1}{2}(\text{surface area of sphere}).$$

Thus, we have

$$2\alpha R^2 + 2\beta R^2 + 2\gamma R^2 - 2\text{Area}(\triangle ABC) = 2\pi R^2.$$

Solving for the area of $\triangle ABC$ gives $\text{Area}(\triangle ABC) = (\alpha + \beta + \gamma - \pi)R^2$, as desired.

The quantity $(\alpha + \beta + \gamma - \pi)$ is referred to as the **spherical excess** of the triangle. Notice that, in order for the area of a triangle to be a positive number, the sum of the angles in a spherical triangle must be greater than $\pi$ radians. Also, recall that by definition, every angle of a spherical triangle is less than two right angles. This means that the sum of the angles in a spherical triangle must be less than six right angles, or $3\pi$ radians. Combining these upper and lower bounds on the angle sum gives the following corollary to the previous theorem.

**Corollary 4.9.** The angle sum for every spherical triangle lies strictly between $\pi$ and $3\pi$ radians.

Let’s extend our definition of a triangle to a general polygon and see what we can determine about its area. We define a **spherical polygon**, or **spherical n-gon**, as an $n$-sided figure (closed and nonself-intersecting) where $n \geq 4$ and the length of each side is less than half of the circumference of a great circle. For example, a **spherical quadrilateral** is a spherical 4-gon. An example of a spherical 8-gon, or octagon, is shown in Figure 4.14. As is the case in Euclidean geometry, angles in an $n$-gon can be less than or greater than $\pi$, but not equal to $\pi$.

![Figure 4.14. Spherical octagon](image)

Recall, in planar geometry, a polygon is called **convex** if the line segment joining any two points of the figure lies entirely in the figure. In Figure 4.15, $ABCDE$ is convex but $FGHIJ$ is not, as demonstrated by its dashed line. Is the octagon shown in Figure 4.14 convex in Spherical geometry? As with the spherical triangle in Section 4.2, we encounter the problem of identifying the interior of an $n$-gon on a sphere since it creates
two trapped regions. In Figure 4.14, for example, we see both a smaller and a larger region trapped by the 8-gon. The smaller region is convex since the line segments (the minor arcs) between any two points on the octagon will stay within the smaller region. This is not true for the larger region since $AC$, for example, does not live in the larger region. Next, let’s consider the vertex angles. Notice that for the smaller octagonal region the vertex angles are all less than $\pi$ radians, whereas, in the larger octagonal region the vertex angles are all greater than $\pi$. Also, the smaller region lives strictly within a hemisphere (all points live in the same hemisphere, as defined by some great circle), whereas the larger region does not. Figure 4.16 shows a 6-gon for which neither the smaller nor the larger trapped region is convex. While the smaller region lives strictly within a hemisphere, its vertex angles are a mix of angles both less than $\pi$ and greater than $\pi$. In general, it can be shown that the polygonal region trapped by a spherical polygon is convex if and only if every vertex angle is less than $\pi$ radians. Furthermore, any convex polygon lives strictly within a hemisphere, with at most one side lying on its boundary. As a consequence, no two points lying either on the boundary or within the figure can be antipodal. We finish this section by calculating the area of a polygonal region meeting this convexity condition, then consider the area of its larger complementary region.

Although we have not formally discussed the axioms of area, we note that one way to calculate the area of a figure is to break it up into smaller figures of known area, then sum the areas of these component pieces. (We will revisit this idea more formally in
In the following proof, we decompose a given quadrilateral region into two spherical triangles in order to determine its area.

**Theorem 4.10.** On a sphere of radius \( R = 1 \), the area of the spherical quadrilateral region, \( Q \), formed by quadrilateral \( ABCD \) with vertex angles \( \alpha, \beta, \gamma \) and \( \delta \), where \( 0 < \alpha, \beta, \gamma, \delta < \pi \), is given by

\[
\text{Area}(Q) = (\alpha + \beta + \gamma + \delta) - 2\pi.
\]

**Proof.** Let quadrilateral region, \( Q \), formed by \( ABCD \), be given as specified, and join \( AC \). Since all vertex angles are less than \( \pi \), \( Q \) is convex. Thus, \( AC \) lies entirely inside our quadrilateral region and, since \( R = 1 \), is less than \( \pi \) in length. Consider the two trilateral regions, \( ABC \) and \( ACD \): each side is less than \( \pi \) by hypothesis and convexity, and every angle is less than \( \pi \) radians by hypothesis and Common Notion 5. Thus, these regions are spherical triangles, \( \triangle ABC \) and \( \triangle ACD \). By Theorem 4.8, \( \text{Area}(\triangle ABC) = (\angle BAC + \beta + \angle BCA) - \pi \) and \( \text{Area}(\triangle ACD) = (\angle CAD + \gamma + \angle ACD) - \pi \). Adding these to find the area of quadrilateral region \( Q \), we have \( \text{Area}(Q) = \text{Area}(\triangle ABC) + \text{Area}(\triangle ACD) = (\angle BAC + \angle CAD + \beta + \angle BCA + \angle CAD + \gamma) - 2\pi = (\alpha + \beta + \gamma + \delta) - 2\pi \), as desired.

We can generalize Theorem 4.10 to the following result, the proof of which we leave to the reader.

**Theorem 4.11.** On a sphere of radius \( R = 1 \), the area of the spherical polygonal region, \( P \), formed by \( n \)-gon \( A_1A_2 \ldots A_n \) with vertex angles \( \alpha_1, \alpha_2, \ldots, \alpha_n \), where each \( \alpha_i < \pi \), is given by

\[
\text{Area}(P) = \left( \sum_{i=1}^{n} \alpha_i \right) - (n - 2)\pi.
\]

When \( P \) is a convex polygonal region as specified in the previous theorem, then the sum of the angles must be less than \( n\pi \). Thus, by Theorem 4.11, the area of \( P \) is less than \( 2\pi \). Since the area of a sphere of radius \( R = 1 \) is \( 4\pi \), we can determine the area of the complement of this region, denoted by \( \overline{P} \), by subtracting the area of \( P \) from the sphere’s area. We also note that \( \overline{P} \) is the polygonal region given by the same vertices, \( A_1, A_2, \ldots, A_n \), but with angles \( \beta_i = 2\pi - \alpha_i > \pi \), since each \( \alpha_i < \pi \). The area of \( \overline{P} \) is calculated as follows:

\[
\text{Area}(\overline{P}) = 4\pi - \text{area}(P)
\]

\[
= 4\pi - \left[ \left( \sum_{i=1}^{n} \alpha_i \right) - (n - 2)\pi \right]
\]

\[
= (n + 2)\pi - \sum_{i=1}^{n} \alpha_i.
\]
Since each angle $\alpha_i < \pi$, the area of $\overline{P}$ is greater than $2\pi$. Of course, we already knew this since these regions are complements of each other with areas that sum to $4\pi$. Notice further, that we can rewrite this area equation as follows:

$$\text{area}(\overline{P}) = \left(2n\pi - \sum_{i=1}^{n} \alpha_i\right) - (n - 2)\pi$$

$$= \left(\sum_{i=1}^{n} (2\pi - \alpha_i)\right) - (n - 2)\pi$$

$$= \left(\sum_{i=1}^{n} \beta_i\right) - (n - 2)\pi.$$

Though $\overline{P}$ is not convex, the formula for its area matches the area formula for a convex polygonal region as given in Theorem 4.11! So, the area equation holds regardless of whether all vertex angles are less than $\pi$, or all are greater than $\pi$. Thus, we have our final result for spherical area.

**Theorem 4.12.** On a sphere of radius $R = 1$, the area of the spherical polygonal region, $P$, formed by $n$-gon $A_1A_2 ... A_n$ with vertex angles $\theta_1, \theta_2, ..., \theta_n$, where either $\theta_i < \pi$ for all $1 \leq i \leq n$, or $\theta_i > \pi$ for all $1 \leq i \leq n$, is given by

$$\text{Area}(P) = \left(\sum_{i=1}^{n} \theta_i\right) - (n - 2)\pi.$$

**Exercises 4.5**

1. Explain why, on a sphere with $R = 1$, the sum of the angles in a triangle that contains at least one right angle must be less than $2\pi$.

2. Use the results of Section 4.5 to explain why, given our definition, there are no rectangles in spherical geometry. Be very specific.

3. Prove Theorem 4.11. [Hint: Decompose the polygonal region by joining one vertex to all nonadjacent vertices.]

4. A **regular spherical polygon** is an equilateral, equiangular spherical polygon.

   (a) Determine the lower bound for the interior angle of a regular spherical $n$-gon whose interior angle is less than $\pi$. [For example, in the case of of a regular spherical pentagon, the angle $\alpha$ must satisfy the inequality $\frac{3\pi}{5} < \alpha < \pi$.]

   (b) Determine the interior angle of a regular Euclidean $n$-gon.

   (c) How does the lower bound for the interior angle of a regular spherical $n$-gon in part (a) relate to the interior angle of a regular Euclidean $n$-gon in part (b)?

### 4.6 Trigonometry for spherical triangles

This next section is a bit of a departure from our geometric discussion as we take a short trip through spherical trigonometry to present three well-known trigonometric formulas that hold for triangles on a sphere. Known as the First Spherical Law of Cosines...
4.6 Trigonometry for spherical triangles

4.6.1 Trigonometry for spherical triangles

The Second Spherical Law of Cosines \([C_2]\), and the Spherical Law of Sines \([S]\), they are given below. Since we will be measuring lengths of segments on a sphere, we need to know the radius, \(R\), of the sphere. It is most convenient to assume that \(R = 1\), although the formulas can be appropriately adjusted if this is not the case. For ease, we assume that we are working on a sphere of radius one unit for the remainder of this chapter unless explicitly stated otherwise.

Let \(\triangle ABC\) be a spherical triangle with sides \(a, b, c\) and interior angles \(\alpha, \beta, \gamma\) as shown in the figure. The laws are given by:

\[
[C_1] \quad \cos a = \cos b \cos c + \cos \alpha \sin b \sin c \\
[C_2] \quad \cos \alpha = \cos a \sin \beta \sin \gamma - \cos \beta \cos \gamma \\
[S] \quad \frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}
\]

In the special case where \(\alpha\) is a right angle, \([C_1]\) simplifies to

\[\cos a = \cos b \cos c.\]

This particular formula is often referred to as the Spherical Pythagorean Theorem. Since it is helpful to have the First and Second Spherical Laws of Cosine solved for both the cosine of an angle and a side, here are their alternative versions:

\[
[C_1^*] \quad \cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \\
[C_2^*] \quad \cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}.
\]

Because lengths are not measured in degrees, it is best to work in radians for the angles in these formulas. If angle measurement is given in degrees, use the conversion factor \(\pi\) radians = 180° to convert to radians before applying the laws. Since we started with the assumption that the radius of the sphere is one unit, these formulas also only work when \(R = 1\). When \(R \neq 1\), simply divide the given lengths, \(a, b\) and \(c\), by \(R\) before applying the laws. When solving, rewrite these formulas (just as we would the Euclidean Law of Cosines) depending upon which pieces of information are given and which are unknown. For example, an alternative version of the First Spherical Law of Cosines, \([C_1^*]\), is

\[\cos \beta = \frac{\cos b - \cos a \cos c}{\sin a \sin c}.\]

When the measures of certain angles or sides are known, we can use the formulas to find the unknowns. If we have a spherical triangle and are given two sides and the included angle, for example \(b, c\) and \(\alpha\), then we can solve for the third side, \(a\), by applying law \([C_1]\). Then, by rewriting \([C_1^*]\) we can solve for \(\beta\) and \(\gamma\), thereby completely determining the triangle, as the following example illustrates.

**Example 4.13.** Suppose we are given angle \(\alpha\), and two sides \(b\) and \(c\), of \(\triangle ABC\) as \(\alpha = \frac{\pi}{2}, b = \frac{2\pi}{3}\) and \(c = \frac{\pi}{2}\). To find the remaining angles and side of the triangle, using
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[Example 4.13 revisited]. Suppose we are given three sides of spherical 
\( \triangle ABC \) as \( a = \frac{\pi}{2} \), \( b = \frac{2\pi}{3} \) and \( c = \frac{\pi}{2} \). To find the angles, using \([C_1]\) we have \( \cos \alpha = 0 \) which gives \( \alpha = \frac{\pi}{2} \). We could continue to use the Law of Cosines to find the remaining unknowns as we did in the previous example. If instead, we use the Law of Sines to find \( \beta \), then we start with the equation \( \frac{1}{1} = \frac{\sin \beta}{\sqrt{3}} \). So, \( \sin \beta = \frac{\sqrt{3}}{2} \), and the arcsine function returns angle \( \frac{\pi}{3} \) for \( \beta \). However, by Proposition I.18, the greater side subtends the greater angle. In our given triangle, since \( b > a \), we know \( \beta > \alpha = \frac{\pi}{2} \). Therefore, our initial solution, \( \frac{\pi}{3} \), cannot be angle \( \beta \). Where is the problem in our reasoning? As noted above, since arcsine only returns angles between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), we will never obtain an obtuse solution. In the given \( \triangle ABC \), \( \beta \) must be obtuse, and the obtuse angle satisfying \( \sin \beta = \frac{\sqrt{3}}{2} \) is \( \beta = \frac{2\pi}{3} \). The lesson: When \( \theta \) is a solution to the Law of Sines equation, either \( \theta \) or \( \pi - \theta \) will be found in the given spherical triangle, but more work is needed to determine which is correct. Since the range of the arcsine function is \([0, \pi]\), no such additional step is necessary when working with the Law of Cosines.

Continuing our exploration of congruence schemes, if we are given two angles and the included side, for example \( \beta, \gamma \) and \( a \), then we can find the third angle, \( \alpha \), by

\[ \cos \alpha = -\frac{1}{2} \cdot 0 + 0, \] which gives \( \alpha = \frac{\pi}{2} \). Using the alternative version of \([C_1]\) as above gives \( \cos \beta = \frac{-\frac{1}{2} - 0}{1 - 1} = -\frac{1}{2} \). So, \( \beta = \frac{2\pi}{3} \). Finally, \( \cos \gamma = 0 \) which gives \( \gamma = \frac{\pi}{2} \). (If, for example, we start with \( \triangle A'B'C' \) on a sphere of radius \( R = 2 \), where \( \alpha' = \frac{\pi}{2} \), \( b' = \frac{4\pi}{3} \) and \( c' = \pi \), we first find its corresponding triangle on the unit sphere in order to utilize the formulas given above. To do this, we convert lengths but leave the angles unchanged. Specifically, here we convert \( b' \) and \( c' \) to \( b = \frac{b'}{R} = \frac{2\pi}{3} \) and \( c = \frac{c'}{R} = \frac{\pi}{2} \), respectively. Notice that the triangle on the unit sphere corresponding to \( \triangle A'B'C' \) is identical to the triangle given at the start of this example, \( \triangle ABC \). Hence, \( a' = Ra = \pi \), \( \beta' = \beta = \frac{2\pi}{3} \) and \( \gamma' = \gamma = \frac{\pi}{2} \).

With this example, it is evident that these formulas provide another way to see that the congruence scheme SAS must hold in Spherical geometry, that is, there is only one such triangle given two sides and their included angle. Alternatively, if we are given three sides of a spherical triangle, \( a, b \) and \( c \), then we can find each of the three angles, \( \alpha, \beta \) and \( \gamma \), after rewriting Law \([C_1]\) for each angle. Thus, these formulas give a different justification for congruence scheme SSS in Spherical geometry.

Before discussing other possible scenarios, a word of caution regarding inverse trigonometric functions is in order. Keep in mind that each angle of a spherical triangle lives in the interval \((0, \pi)\), and while the range of the inverse cosine function is \([0, \pi]\), the range of the inverse sine function is \([-\frac{\pi}{2}, \frac{\pi}{2}]\). Therefore, as we shall see in the next example, solutions obtained by employing the arcsine function must be verified since this function never produces an obtuse angle.

**Example 4.14** (Example 4.13 revisited). Suppose we are given three sides of spherical \( \triangle ABC \) as \( a = \frac{\pi}{2} \), \( b = \frac{2\pi}{3} \) and \( c = \frac{\pi}{2} \). To find the angles, using \([C_1]\) we have \( \cos \alpha = 0 \) which gives \( \alpha = \frac{\pi}{2} \). We could continue to use the Law of Cosines to find the remaining unknowns as we did in the previous example. If instead, we use the Law of Sines to find \( \beta \), then we start with the equation \( \frac{1}{1} = \frac{\sin \beta}{\sqrt{3}} \). So, \( \sin \beta = \frac{\sqrt{3}}{2} \), and the arccosine function returns angle \( \frac{\pi}{3} \) for \( \beta \). However, by Proposition I.18, the greater side subtends the greater angle. In our given triangle, since \( b > a \), we know \( \beta > \alpha = \frac{\pi}{2} \). Therefore, our initial solution, \( \frac{\pi}{3} \), cannot be angle \( \beta \). Where is the problem in our reasoning? As noted above, since arcsine only returns angles between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), we will never obtain an obtuse solution. In the given \( \triangle ABC \), \( \beta \) must be obtuse, and the obtuse angle satisfying \( \sin \beta = \frac{\sqrt{3}}{2} \) is \( \beta = \frac{2\pi}{3} \). The lesson: When \( \theta \) is a solution to the Law of Sines equation, either \( \theta \) or \( \pi - \theta \) will be found in the given spherical triangle, but more work is needed to determine which is correct. Since the range of the arccosine function is \([0, \pi]\), no such additional step is necessary when working with the Law of Cosines.
applying Law \([C_2]\). Then, by using \([C_2]\), we can solve for sides \(b\) and \(c\). Thus, ASA must hold. This line of reasoning leads us to a fourth congruence scheme, one that is not valid in Euclidean geometry. If we know \(\alpha, \beta\) and \(\gamma\), then by rewriting and using \([C_2]\) we can solve for sides \(a, b\) and \(c\). Thus, in Spherical geometry, AAA is a valid congruence scheme. Exercise 4.6.1 demonstrates this congruence scheme as there is a unique triangle associated with any given set of three angles. We will state this as a theorem and give its proof in Section 4.7.

With these laws providing justification to four triangle congruence schemes on the sphere, the temptation to extend this line of reasoning to the unexplored AAS and SSA schemes is strong. How about AAS? In Section 4.4, we noted that this scheme does not hold in Spherical geometry and asked the reader to provide a counterexample. Does the trigonometry support our earlier work? Suppose we are given \(\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}\) and \(a = \frac{\pi}{2}\) for \(\triangle ABC\). The Law of Sines gives \(b = \frac{\pi}{2}\), and \([C_1]\) gives \(\cos \gamma = \cos c\). Thus, while we know \(\gamma = c\), we do not have enough information to solve for this last unknown. Try \(c = \frac{\pi}{3}\) and \(c = \frac{2\pi}{3}\) for just two of the infinitely many possible values of \(c\) that satisfy these equations. Thus, unlike Euclidean geometry, two angles and an unincluded side is not enough information to determine a spherical triangle. The exploration of the possibility of SSA as a triangle congruence scheme on the sphere is left as an exercise.

For our final investigation, we revisit the restriction placed on constructible equilateral triangles in Proposition I.1S. First, since the total angle sum for spherical triangles ranges strictly between \(\pi\) and \(3\pi\), it is clear that the interior angle of a spherical equilateral triangle ranges strictly between \(\frac{\pi}{3}\) and \(\pi\) by Corollary 4.9. Its Euclidean counterpart, however, is fixed at \(\frac{\pi}{3}\). The Law of Cosines tells us how the side of an equilateral triangle is related to its angle. Assuming that the triangle has side of length \(a\) and angle \(\alpha\), Law \([C_2]\) gives

\[
\cos a = \frac{\cos \alpha + \cos^2 \alpha}{\sin^2 \alpha} = \frac{\cos \alpha (1 - \cos \alpha)}{1 - \cos^2 \alpha} = \frac{\cos \alpha}{1 - \cos \alpha}.
\]

As shown in Figure 4.17, the graph of function \(a = \arccos \left( \frac{\cos \alpha}{1 - \cos \alpha} \right)\) over the domain of \(\frac{\pi}{3} < \alpha < \pi\) has a range of \(0 < a < \frac{2\pi}{3}\). Thus, the side length is necessarily less than
\( \frac{2\pi}{3} \), or one-third of the circumference of a great circle. This explains why the circles in the proof of Proposition 1.1 do not intersect when the given segment has length at least one-third of the circumference of a great circle, as clearly, no such spherical equilateral triangle exists. Hence, we have justified the additional restriction placed in Proposition 1.1.

Exercises 4.6

1. On a sphere of radius \( R = 1 \), solve the spherical triangle with angles:
   (a) \( \alpha = 60^\circ, \beta = 60^\circ, \gamma = 90^\circ \)
   (b) \( \alpha = 60^\circ, \beta = 90^\circ, \gamma = 90^\circ \)
   (c) \( \alpha = 90^\circ, \beta = 90^\circ, \gamma = 90^\circ \)
   (d) \( \alpha = 60^\circ, \beta = 90^\circ, \gamma = 120^\circ \)

2. On a sphere of radius \( R = 1 \), solve the spherical triangle with sides:
   (a) \( a = \frac{\pi}{2}, b = \frac{\pi}{2}, c = \frac{\pi}{2} \)
   (b) \( a = 1, b = 1, c = 1 \)
   (c) \( a = \frac{\pi}{6}, b = \frac{\pi}{4}, c = \frac{\pi}{4} \)
   (d) \( a = \frac{\pi}{4}, b = \frac{\pi}{3}, c = \frac{\pi}{2} \)

3. On a sphere of radius \( R = 1 \), solve the spherical triangle with:
   (a) \( a = \frac{\pi}{2}, \beta = 60^\circ, \gamma = 60^\circ \)
   (b) \( b = \frac{\pi}{4}, \alpha = 90^\circ, \gamma = 45^\circ \)
   (c) \( c = \frac{\pi}{2}, \alpha = 60^\circ, \beta = 120^\circ \)
   (d) \( a = \frac{\pi}{2}, \beta = 45^\circ, \gamma = 135^\circ \)

4. On a sphere of radius \( R = 1 \), solve the spherical triangle with:
   (a) \( a = \frac{\pi}{2}, b = \frac{\pi}{2}, \gamma = 60^\circ \)
   (b) \( a = \frac{\pi}{4}, b = \frac{\pi}{2}, \gamma = 90^\circ \)
   (c) \( b = \frac{\pi}{4}, c = \frac{3\pi}{4}, \alpha = 60^\circ \)

5. On a sphere of radius \( R \), solve the spherical triangle:
   (a) with sides \( R, R, R \)
   (b) with \( b = 2R, \alpha = \frac{\pi}{3}, \gamma = \frac{\pi}{2} \)

6. Calculate the area of each spherical triangle given in the first five exercises.

7. Explore the possibility of an SSA triangle congruence scheme on the sphere: Suppose \( a = \frac{\pi}{2}, b = \frac{\pi}{2} \) and \( \alpha = \frac{\pi}{2} \) for \( \triangle ABC \).
   (a) Use the Laws of Cosines and Sines to determine if this information is sufficient to determine triangle \( \triangle ABC \). If the answer is yes, then prove it. If the answer is no, then provide two different triangles with the given \( a, b \) and \( \alpha \).
(b) Is SSA a valid congruence scheme in Spherical geometry?

8. Given an equilateral triangle with angle $\alpha$ and side length $a$, give a formula for $\cos \alpha$ in terms of $\cos a$.

9. Prove the following proposition: “If two angles of a spherical triangle are supplements of each other, their opposite sides are also supplements of each other” [112].

10. “Find the angles and sides of an equilateral spherical triangle whose area is one-fourth of that of the sphere on which it is described” [117]. Note: Assume we are working on a sphere of radius $R = 1$.

11. In an equilateral spherical triangle with angle $\alpha$ and side $a$, show that

$$\sec \alpha = 1 + \sec a.$$ 

12. In a spherical triangle with sides $a$, $b$ and $c$ and opposite angles $\alpha$, $\beta$ and $\gamma$, respectively, if $a = \alpha$, show that either $b = \beta$ or $b = \pi - \beta$, and either $c = \gamma$ or $c = \pi - \gamma$. That is, $b$ and $\beta$ are equal or supplemental, as are $c$ and $\gamma$.

13. “If two angles of a spherical triangle be respectively equal to the sides opposite them, show that the remaining side is the supplement of the remaining angle; or else that the triangle has two quadrants (sides with lengths equal to one-quarter of the circumference) and two right angles, and then the remaining side is equal to the remaining angle” [117]. In other words, in a spherical triangle with sides $a$, $b$ and $c$ and opposite angles $\alpha$, $\beta$ and $\gamma$, respectively, assume that $a = \alpha$ and $b = \beta$. Show that if $a = b = \pi/2$, then $c = \gamma$, otherwise $c = \pi - \gamma$.

14. In a spherical triangle with sides $a$, $b$ and $c$ and opposite angles $\alpha$, $\beta$ and $\gamma$, respectively, if $b + c = \pi$, show that $\sin 2\beta + \sin 2\gamma = 0$.

15. Consider the spherical triangle $\triangle ABC$ with $a = BC$, $b = AC$ and $c = AB$. Let $D$ be the midpoint of $AB$, and let $d = CD$, as shown in Figure 4.18. Prove that

$$\cos a + \cos b = 2 \cos \frac{c}{2} \cos d.$$ 

![Figure 4.18. Exercise 4.6.15](image)

16. “A triangle has the sum of two sides equal to a semicircumference: find the arc joining the vertex with the middle of the base” [117]. In other words, consider the spherical triangle $\triangle ABC$ with $a = BC$, $b = AC$ and $c = AB$. Let $D$ be the midpoint of $AB$, and let $d = CD$, as shown in Figure 4.18. Since the semicircumference is $\pi$ when

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6Exercises 4.6.10 through 4.6.17 are adapted from Todhunter’s nineteenth century spherical trigonometry book, [117].
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\[ R = 1 \], we assume \( a + b = \pi \), and we must find \( d \). [Hint: Use the formula in Exercise 15.]

17. Consider the spherical triangle \( \triangle ABC \). Let \( D \) and \( E \) be the midpoints of \( AB \) and \( AC \), respectively. Join \( DE \) and let \( F \) be a polar point for \( DE \). Join \( FB, FD, FE \), and \( FC \). Show that \( \angle BFC = 2\angle DFE \). [Hint: Join \( AF \) and then show that \( \angle AFD = \angle BFD \) and \( \angle AFE = \angle CFE \).]

### 4.7 Uniquely spherical constructions

In Chapter 3, we encountered the ten constructions of Neutral geometry, representing just over one-third of its propositions. In Section 4.4, our analysis of these constructions on the sphere showed that six of these are valid, namely Propositions I.3, I.9, I.10, I.11, I.12 and I.23, though some require modification to Euclid’s proof; two are not valid, namely I.1 and I.22, unless a restriction is placed upon the length of at least one of the given segments; two are invalid, namely I.2 (which was added to the postulates) and I.31 (since parallel lines do not exist on the sphere). In this final section of the chapter, we explore uniquely Spherical constructions resulting from the close relationship that exists between lengths and angles on the sphere. Don’t try these in Euclidean geometry! Here, we continue to assume that we are working on a unit sphere \( (R = 1) \) and our angle measure is in radians. We start with two somewhat surprising results that allow us to convert a given length to an angle, or vice versa.

**Theorem 4.15.** To construct a length equal to a given angle \( \alpha \), where \( \alpha < \pi \).

*Proof.* Let \( \angle BAC = \alpha \). Extend \( AB \) and \( AC \) until they intersect at \( A' \) which is antipodal to \( A \). Bisect \( \angle B\hat{A}C \) and \( \angle C\hat{A}B \). Label these points \( D \) and \( E \), respectively, and construct segment \( DE \). We claim that \( DE = \alpha \). Consider the line containing \( DE \). By definition, \( A \) is a polar point of \( DE \). Note that the total angle surrounding \( A \) is \( 2\pi \). Similarly, if we extend segment \( DE \) to form great circle \( DE' \), it too has a length of \( 2\pi \). Thus

\[
\frac{\alpha}{2\pi} = \frac{DE}{2\pi},
\]

and \( \alpha = DE \), as desired. \( \square \)

**Theorem 4.16.** To construct an angle equal to a given length less than half of the circumference.

We leave the proof of the second theorem, which is similar to the previous, as an exercise. Thus, we can convert angles to lengths, and vice versa.

Continuing down this path of exploration, we would like to construct a new triangle from a given triangle in such a way that the angles of the new triangle are based on the sides of the given triangle, and vice versa. This construction produces a new object called a polar triangle which was introduced by Abu Nasr Mansur ibn Iraq (ca. 970–ca. 1036), an astronomer and mathematician who worked in Ghazna in present-day Afghanistan [121].

**Definition 4.17.** Let \( \triangle ABC \) be a spherical triangle. Let \( C' \) be the polar point of \( AB \) that lies in the same hemisphere as \( C \). In a similar manner, construct points \( A' \) and \( B' \).
4.7 Uniquely spherical constructions

Points $A'$, $B'$ and $C'$ are polar points of sides $BC$, $AC$ and $AB$, respectively. We will call $\triangle A'B'C'$ the polar triangle of $\triangle ABC$.

Although it is far from obvious, there is a very close relationship between a triangle and its associated polar triangle. This relationship is best described by two well-known theorems, the first of which shows that the process of constructing a polar triangle is an involution, that is, it is its own inverse.

**Theorem 4.18 [Polar Triangle Involution Theorem].** The polar triangle of a polar triangle is the original triangle.

*Proof.* Let $\triangle ABC$ be a given spherical triangle and $\triangle A'B'C'$ be its polar triangle. Since $C'$ is a pole of $AB$ and $A'$ is a pole of $BC$, both $A'$ and $C'$ are a length of $\frac{\pi}{2}$ from $B$. Thus $B$ is a pole of $A'C'$.

We will now show that $B$ is on the same side of $A'C'$ as $B'$. Since $B'$ is a pole of $AC$ on the same side as $B$, the distance between $B$ and $B'$ must be less than one-quarter of the circumference. If we assume that $B$ is not on the same side of $A'C'$ as $B'$, then the antipodal point of $B$, which we will call $B''$, must be on the same side of $A'C'$ as $B'$. But this would imply that the distance between $B'$ and $B''$ is also less than one-quarter of the circumference of a great circle, which is not possible as the distance between antipodal points $B$ and $B''$ is exactly half of the circumference, and $B'$ lies on a unique line between them. Therefore $B$ and $B'$ must be on the same side of $A'C'$.

By similar reasoning, $A$ and $C$ are the appropriate polar points for $B'C''$ and $A'B'$, respectively.

**Theorem 4.19 [Polar Duality Theorem].** The sides of a polar triangle are the supplements of the angles of the original triangle, and the angles of a polar triangle are the supplements of the sides of the original. [Note that the supplement of side $a$ is $\pi - a$.]

*Proof.* Let $\triangle ABC$ be a given spherical triangle with angles $\alpha$, $\beta$ and $\gamma$ at vertices $A$, $B$ and $C$, respectively. Consider its polar triangle $\triangle A'B'C'$. We will start by showing that $B'C'$ has length $\pi - \alpha$. As demonstrated in Figure 4.20, extend rays $\overline{AB}$ and $\overline{AC}$ so that the first intersection with the line containing $B'C'$ are points $D$ and $E$, respectively.
Figure 4.20. Polar Duality Theorem

Notice that $\angle DAE = \alpha$. Since $A$ is a polar point for the line $B'C'$, by Theorem 4.15 we have $DE = \alpha$. Furthermore, since $C'$ is a polar point for the line $AB$, and $D$ lies on $AB$, we have $DC' = \pi/2$. By similar reasoning, $BE = \pi/2$. We leave it to the reader to show that $DE$ cannot straddle $B'C'$, leaving only two possible cases.

**Case 1.** $B'C'$ contains segment $DE$. In this case, we have the four collinear points as specified by $B'DEC'$. Thus $B'C' = B'E + DC' - DE = \pi/2 + \pi/2 - \alpha = \pi - \alpha$, as desired.

**Case 2.** $DE$ contains segment $B'C'$, as illustrated in Figure 4.20. In this case, we have collinear points as specified by $DB'C'E$. Thus, once again, $B'C' = DC' + B'E - DE = \pi/2 + \pi/2 - \alpha = \pi - \alpha$.

By similar reasoning, we have $A'B' = \pi - \gamma$ and $A'C' = \pi - \beta$. To show that the angles are supplements of the sides of the original, we can apply the results we have just proven along with Theorem 4.18 to polar triangle $\triangle A'B'C'$.

Notice that while our name for this useful trilateral figure corresponding to a spherical triangle is polar triangle, we are only now guaranteed that a polar triangle is a spherical triangle. By the Polar Duality Theorem, when $\triangle ABC$ has sides $a = \pi/6$, $b = \pi/4$ and $c = \pi/3$, for example, its polar triangle has angles $\alpha' = 5\pi/6, \beta' = 3\pi/4$ and $\gamma' = 2\pi/3$.

You may recall that Lemma 4.5 guarantees that the sum of the sides of a spherical triangle is less than the circumference of a great circle, or, in the case of a unit sphere, $2\pi$. The Polar Duality Theorem provides an alternative proof for this lemma as follows. Given $\triangle ABC$ with sides $a, b$ and $c$, consider its polar triangle $\triangle A'B'C'$. By the Polar Duality Theorem, $\triangle A'B'C'$ has angles $\alpha' = \pi - a, \beta' = \pi - b$ and $\gamma' = \pi - c$. Adding the angles of $\triangle A'B'C'$ gives

$$\alpha' + \beta' + \gamma' = 3\pi - (a + b + c).$$

Thus, we have

$$a + b + c = 3\pi - (\alpha' + \beta' + \gamma').$$

By Corollary 4.9, the sum of the angles of $\triangle A'B'C'$ is larger than $\pi$. Therefore, $a + b + c < 2\pi$, and the total length of the sides for any spherical triangle is less than $2\pi$. 
4.7 Uniquely spherical constructions

When combined with the Triangle Inequality, the duality theorem also determines a condition on the angles of a spherical triangle. Specifically, since any two sides taken together must be greater than the third, by the Polar Duality Theorem we have that the sum of any two angles must be less than the remaining angle augmented by $\pi$, for example, $\alpha + \beta < \pi + \gamma$.

**Corollary 4.20.** In a spherical triangle, the sum of any two angles is less than the remaining angle augmented by $\pi$.

With this corollary, it is easy to see that while there are fewer restrictions on the angles of a spherical triangle than there are in Euclidean geometry, there are still sets of angles whose sum lies between $\pi$ and $3\pi$ for which there is no corresponding spherical triangle. For example, if we let $\alpha = \frac{3\pi}{4}, \beta = \frac{3\pi}{4}$ and $\gamma = \frac{\pi}{4}$, then $\alpha + \beta < \pi + \gamma$. Therefore, a spherical triangle with these angles does not exist. The Law of Cosines, $[C^2_2]$, further supports this, giving

$$\cos \alpha = \frac{-\sqrt{2}}{2} + \left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - 1 < -1.$$ 

Clearly, there is no side $a$ which satisfies this equation.

When a triangle with a given set of angles does exist, we can use the Polar Duality Theorem in conjunction with the close relationship between lengths and angles established in this section to construct a triangle with the given angles. We give this construction as the following theorem.

**Theorem 4.21.** Given three angles between 0 and $\pi$ whose sum is greater than $\pi$, and such that the sum of any two angles is less than the remaining angle augmented by $\pi$, to construct a triangle.

**Proof.** Let $\alpha, \beta$ and $\gamma$ be the given angles. By assumption, $\alpha, \beta, \gamma < \pi$. By Theorem 4.15, construct a segment with length $\alpha$ and label its endpoints $A$ and $B$. Construct $A'$, the antipodal point to $A$. Segment $A'B$ has length $a' = \pi - \alpha$, which is less than half the circumference of a great circle. Similarly, construct segments of length $b' = \pi - \beta$ and $c' = \pi - \gamma$. Because the sum of any two angles is less than the remaining angle augmented by $\pi$, the lengths $a', b'$ and $c'$ satisfy the Triangle Inequality, namely, that any two sides taken together will be greater than the third. Moreover, since $\alpha + \beta + \gamma > \pi$, we have $a' + b' + c' < 2\pi$. Using Proposition II.22S, construct triangle $\triangle CDE$ with sides of these lengths. By the Polar Duality Theorem, the polar triangle of $\triangle CDE$ is the desired triangle since it has the given angles $\alpha, \beta$ and $\gamma$.

The final congruence scheme for spherical triangles, AAA$_S$, follows directly from the theorem. We will refer to this theorem as AAA$_S$, with the subscript $S$ as a reminder that it is a congruence scheme for Spherical geometry.

**Corollary 4.22 [AAA$_S$].** Two spherical triangles with equal angles are congruent.

Before leaving this world, we take one last look at equilateral triangles. Note that the polar triangle of an equilateral triangle is equilateral. As discussed in Section 4.6,
the angle, \( \alpha \), of an equilateral triangle on a sphere lies between \( \pi/3 \) and \( \pi \). Given such an \( \alpha \), we can construct the equiangular triangle with this angle using Theorem 4.21. Thus, by polar duality, an equilateral triangle must have side length strictly between 0 and \( 2\pi/3 \), giving yet another justification for Proposition I.1S.

We are now ready to leave the spherical world, where angles behave in unexpected ways, to explore a very different world where distances behave in curious ways.

**Exercises 4.7**

1. Prove Theorem 4.16: *To construct an angle equal to a given length less than half of the circumference.*

2. Prove that in the proof of Theorem 4.19 the only possible cases are that either \( B'C' \) contains \( DE \) or \( DE \) contains \( B'C' \).

3. Determine when a triangle and its polar triangle coincide.

4. Consider the triangle \( \triangle ABC \), where \( a = b = \pi/2 \) and \( c = \pi/4 \). Determine the angles and sides of its polar triangle.